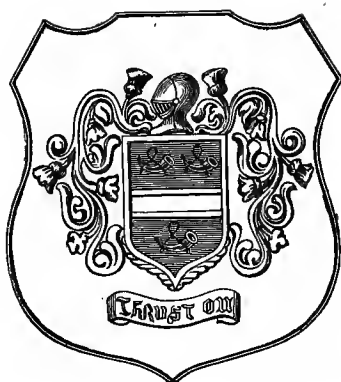
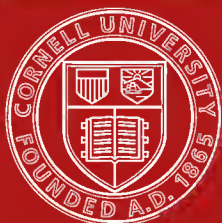




Robert Henry Thurston

A Gift to
Cornell University Library
1903



Cornell University
Library

The original of this book is in
the Cornell University Library.

There are no known copyright restrictions in
the United States on the use of the text.

THEORETICAL MECHANICS



AN
ELEMENTARY TREATISE
ON
THEORETICAL MECHANICS

BY
ALEXANDER ZIWET
ASSISTANT PROFESSOR OF MATHEMATICS IN THE
UNIVERSITY OF MICHIGAN

PART II:
INTRODUCTION TO DYNAMICS; STATICS

New York
MACMILLAN AND CO.
AND LONDON
1893

All rights reserved

COPYRIGHT, 1893,
By MACMILLAN & CO.

Notwood Press :
J. S. Cushing & Co.—Berwick & Smith.
Boston, Mass., U.S.A.

PREFACE.



THE subject of statics is here developed only so far as is absolutely necessary in order to lay the foundation on the one hand for the study of elementary kinetics, on the other for applied mechanics. From the former point of view it was desirable to bring out clearly the connection of the subject with the general science of mechanics and to determine its place as a subdivision of the larger science. The second section of Chapter III should be considered only as preliminary; the fundamental laws of dynamics can of course be fully understood only by studying kinetics. Prominence is given throughout to geometrical methods and graphical constructions because these seem to conform best to the nature of the subject. The applications given here and there are to be regarded merely as illustrations of the general principles.

The following articles might be omitted at first reading: 18, 19, 20, 34, 43, 44, 48, 52, 113, 117-127, 152-164, 180, 181, 209, 210, 214, 220-225, 257-285.

ALEXANDER ZIWET.

ANN ARBOR, MICH.,
October, 1893.

CORRIGENDA TO PART I.

Page 16, line 11 from the bottom, for “ l ” read “ l' .”

Page 19, line 11 from the bottom, after “lines” insert “and one point.”

Page 29, line 8 from the top, for “50” read “25.”

Page 62, line 7 from the top, for “ s ” read “ s^2 .”

Page 62, line 9 from the top, for “ $-\frac{1}{2}\mu^2 R$ ” read “ $-\frac{1}{2}\mu^2 R^2$.”

Page 74, line 6 from the top, for “6 minutes” read “6 seconds.”

Page 77, line 4 from the bottom, for “ j ” read “ a .”

Page 176, line 8 from the bottom, for “ $\frac{1}{3}$ ” read “ $\frac{1}{10}$.”

ge 177, line 10 from the top, for “13 m. $32\frac{1}{2}$ s.” read “34 m. 48 s.”

Page 177, lines 13-15: The answer to (5) should read:

$$\begin{aligned} \text{“ Height} = R; \text{ time of ascent} &= \sqrt{\frac{R}{g}} \left(1 + \frac{\pi}{2} \right) = \text{time of falling back} \\ &= 34 \text{ m. 48 s.} \end{aligned}$$

Page 177, line 18, for “2602” read “26 000.”

CONTENTS.



CHAPTER III.

INTRODUCTION TO DYNAMICS.

	PAGE
I. MASS; MOMENTS OF MASS; CENTROIDS:	
1. Mass; density	I
2. Moments and centres of mass	4
3. Examples of the determination of centroids	11
II. MOMENTUM; FORCE; ENERGY	34

CHAPTER IV.

STATICS.

I. INTRODUCTION	45
II. CONCURRENT FORCES	54
III. PARALLEL FORCES	59
IV. THEORY OF COUPLES	76
V. PLANE STATICS:	
1. The conditions of equilibrium	83
2. Stability	92
3. Jointed frames	99
4. Graphical methods	103
5. Friction	109

VI. SOLID STATICS:	PAGE
1. The conditions of equilibrium	121
2. Constraints	132
VII. THE PRINCIPLE OF VIRTUAL WORK	143
VIII. THEORY OF ATTRACTIVE FORCES:	
1. Attraction	158
2. The potential	169
ANSWERS	176

THEORETICAL MECHANICS.



CHAPTER III.

INTRODUCTION TO DYNAMICS.

I. *Mass ; Moments of Mass ; Centroids.*

I. MASS ; DENSITY.

1. In the first part of this work only the geometrical and kinematical properties of motion have been considered, the moving object being regarded as a mere point or as a geometrical configuration. It is, however, known, from observation and experiment, that the motions of actual physical bodies are not fully described and determined by those properties alone.

Physical bodies are distinguished from geometrical configurations by being possessed of **mass**; and this property as affecting their motion must be taken into account in dynamics.

2. In physics the mass of a body is usually defined as *the quantity of matter contained in the body*. Postponing for the present the full discussion of the idea of mass in its relation to acceleration and force, and of the methods for comparing and measuring masses, it will suffice for our present purpose to think of the mass of a body as a certain constant quantity, independent of the body's position or motion with respect to the earth or other bodies, as an indestructible something underlying every physical body.

The student must be warned not to confound mass with weight. The weight of a body, as we shall see later, is the force with which the body is attracted by the earth; it varies,

therefore, with the distance of the body from the earth's centre, and would vanish completely if the earth were suddenly annihilated; while the *indestructibility of mass* is the first fundamental principle of chemistry and physics.

3. To compare the masses of different bodies, we may adopt any given body as a standard.

Thus in the F. P. S. system, the *standard mass* is a certain bar of platinum marked "P. S., 1844, 1 lb.," and preserved at the Office of the Exchequer, London, England. This is called the "imperial standard pound avoirdupois"; any mass equal to it is a *unit of mass* in this system.

In the C. G. S. system, the standard of mass is the "Kilogramme des archives," a bar of platinum kept in the Palais des archives, in Paris, France. A mass equal to one-thousandth of this standard is the unit of mass in this system; this unit is called the *gram*.

The numerical relation between the British and metric units of mass is as follows:

$$1 \text{ lb.} = 453.59265 \text{ gm.}$$

$$1 \text{ gm.} = 0.0022046212 \text{ lb.} = 15.432 \text{ grains.}$$

4. The three units of *space*, *time*, and *mass* are called the *fundamental units of mechanics*, because with the aid of these three, the units of all other quantities occurring in mechanics can be expressed. Thus we have seen how the units of velocity and acceleration are based on those of space and time, and we shall have many more illustrations in what follows. Any unit that can be expressed mathematically by means of one or more of the fundamental units is called a *derived unit*.

5. From the mathematical point of view, mass appears in our dynamical equations as a coefficient, generally to be regarded as an absolute, positive constant. It serves to give different values (different valency, or "weight" in the meaning of the theory of least squares) to the moving points, lines, areas, volumes, apart from their geometrical extension.

6. Thus, a geometrical point endowed with mass is called a *material particle*. We may regard such a mass-point, or particle, as the limit to which a physical body approaches if its volume be imagined to decrease indefinitely, approaching the limit zero, while its mass may remain a finite quantity. From the physical point of view a particle must be regarded as much an abstraction as a geometrical point, since every finite physical mass occupies a finite space and cannot be identified with a point. We shall see, however, that in dynamics this idea of the mass-point, or particle, is of the greatest importance not only because physical matter is usually considered as made up of an aggregation of such points or centres possessing mass (molecules, atoms), but principally because in many cases the motion of a solid body can be fully represented by the motion of a certain point in it, called its *centre of mass* or **centroid**, the whole mass being regarded as concentrated at this point.

7. It is also customary in dynamics to speak of *material lines* and *material surfaces*, which may be regarded as the limits of physical bodies in which two dimensions or one dimension have been reduced to zero. Thus a material line represents the limit of a wire, chain, or bar, in which two dimensions are neglected; a material surface can be imagined as the limit of a thin shell, or lamina, with one dimension reduced to zero.

8. A continuous mass of one, two, or three dimensions, is said to be *homogeneous* if the masses contained in *any* two equal lengths, areas, or volumes (as the case may be), are equal. The mass is then said to be distributed *uniformly*. In all other cases the mass is said to be *heterogeneous*.

9. The whole mass M of a homogeneous body divided by the space V it fills is called the **density** of the mass or body; denoting density by ρ we have therefore

$$\rho = \frac{M}{V},$$

for homogeneous bodies.

In a heterogeneous body, this quotient is called the *average*, or *mean*, density. In this case *the density at any point*, or the density of any space-element dV , is defined as the derivative

$$\rho = \frac{dM}{dV}$$

10. The *unit of density* is the density of a substance such that the unit of volume contains the unit of mass. If the units of volume and mass are selected arbitrarily, there need not of course necessarily exist any physical substance having unit density exactly. Thus in the F.P.S. system, unit density is the density of an ideal substance 1 pound of which would just fill a cubic foot. As a cubic foot of water has a mass of $62\frac{1}{2}$ pounds, or 1000 ounces, the density of water is $62\frac{1}{2}$ times the unit density.

The *specific density*, or *specific gravity*, of a substance, is the ratio of its density to that of water at 4° C. Let ρ be the density, ρ' the specific density, M the mass, V the volume of a homogeneous mass, then in British units

$$M = \rho V = 62.5 \rho' V.$$

In the C.G.S. system, the unit of mass has been so selected as to make the density of water equal to 1 very nearly; in other words, the unit mass (1 gramme) of water, at the temperature of 4° C., fills one cubic centimetre.

In the metric system, then, there is no difference between density and specific density or specific gravity.

2. MOMENTS AND CENTRES OF MASS.

11. The product of a mass m , concentrated at a point P , into the distance of the point P from any given point, line, or plane, is called the **moment** of this mass with respect to the point, line, or plane.

Thus, denoting by r, q, p , the distance of the point P from the point O , the line l , and the plane π , respectively, we have for the moments of m with respect to O, l, π , the expressions mr, mq, mp .

12. Let a system of n points, or particles, $P_1, P_2, \dots P_n$ be given; let $m_1, m_2, \dots m_n$ be their masses, and $p_1, p_2, \dots p_n$ their distances from a given plane π . Then we call **moment of the system** with respect to the plane π the algebraic sum

$$m_1 p_1 + m_2 p_2 + \dots + m_n p_n = \Sigma m p,$$

the distances $p_1, p_2, \dots p_n$ being taken with the same sign or opposite signs according as they lie on the same side or on opposite sides of the plane π .

It is always possible to determine one and only one distance $\bar{p}$ such that $\Sigma m p = M \bar{p}$, where $M = \Sigma m = m_1 + m_2 + \dots + m_n$ is the total mass of the system. If this distance $\bar{p}$ should happen to be equal to zero, the moment of the system would evidently vanish with respect to the plane π .

13. Let us now refer the points P to a rectangular system of co-ordinates, and let x, y, z be their co-ordinates. Then we have for the moments of the system with respect to the co-ordinate planes yz, zx, xy , respectively

$$m_1 x_1 + m_2 x_2 + \dots + m_n x_n = \Sigma m x = M \bar{x},$$

$$m_1 y_1 + m_2 y_2 + \dots + m_n y_n = \Sigma m y = M \bar{y},$$

$$m_1 z_1 + m_2 z_2 + \dots + m_n z_n = \Sigma m z = M \bar{z}.$$

The point G whose co-ordinates are

$$\bar{x} = \frac{\Sigma m x}{M}, \quad \bar{y} = \frac{\Sigma m y}{M}, \quad \bar{z} = \frac{\Sigma m z}{M} \quad (1)$$

is called the *centre of mass*, or the **centroid**, of the system.

The centroid is, therefore, defined as a point such that if the whole mass M of the system be concentrated at this point, its

moment with respect to any one of the co-ordinate planes is equal to the moment of the system.

14. It is easy to see that this holds not only for the co-ordinate planes but for any plane whatever. Let

$$\alpha x + \beta y + \gamma z - p_0 = 0$$

be the equation of any plane in the normal form ;

$$\bar{p}, p_1, p_2, \dots, p_n$$

the distances of the points $G, P_1, P_2, \dots, P_n$ from this plane. Then we wish to prove that $\Sigma m\bar{p} = M\bar{p}$.

$$\text{Now} \quad \bar{p} = \alpha \bar{x} + \beta \bar{y} + \gamma \bar{z} - p_0,$$

$$p_1 = \alpha x_1 + \beta y_1 + \gamma z_1 - p_0, \dots$$

$$\text{hence} \quad \Sigma m\bar{p} = \alpha \Sigma mx + \beta \Sigma my + \gamma \Sigma mz - p_0 \Sigma m$$

$$= M(\alpha \bar{x} + \beta \bar{y} + \gamma \bar{z} - p_0)$$

$$= M\bar{p}.$$

The centroid can therefore be defined as a point such that its moment with respect to any plane is equal to that of the whole system, with respect to the same plane.

It follows that *the moment of the system vanishes for any plane passing through the centroid.*

15. In the case of a continuous mass, whether it be of one, two, or three dimensions, the same reasoning will apply if we imagine the mass divided up into elements dM of one, two, or three infinitesimal dimensions, respectively. The summations indicated above by Σ will then become integrations, so that the centroid of a continuous mass has the co-ordinates

$$\bar{x} = \frac{\int x dM}{\int dM}, \quad \bar{y} = \frac{\int y dM}{\int dM}, \quad \bar{z} = \frac{\int z dM}{\int dM}. \quad (2)$$

According as the mass is of one, two, or three dimensions, a single, double, or triple integration over the whole mass will in general be required for the determination of the moments $\int x dM$, $\int y dM$, $\int z dM$ of the mass with respect to the co-ordinate planes, as well as of the total mass $\int dM = M$.

Thus, for a mass distributed along a line or a curve we have, if ds be the line-element,

$$dM = \rho ds;$$

for a mass distributed over a surface-area we have, with dS as a surface-element,

$$dM = \rho dS;$$

finally, for a mass distributed throughout a volume whose element is dV ,

$$dM = \rho dV.$$

If the mass be distributed along a straight line, the centroid lies of course on this line, and one co-ordinate is sufficient to determine the position of the centroid. In the case of a plane area, the centroid lies in the plane and two co-ordinates determine its position; we then speak of moments with respect to lines, instead of planes.

16. If the mass be homogeneous (Art. 8), *i.e.* if the density ρ be constant, it will be noticed that ρ cancels from the numerator and denominator in the equations (2), and does not enter into the problem. Instead of speaking of a centre of mass, we may then speak of a centre of arc, of area, of volume. The term *centroid* is, however, to be preferred to *centre*, the latter term having a recognised geometrical meaning different from that of the former.

The geometrical centre of a curve or surface is a point such that any chord through it is bisected by the point; there are but few curves and surfaces possessing a centre.

The centroid (Art. 14) is a point such that, for any plane passing through it, the moment of the system is equal to zero. Such a point exists for every mass, volume, area, or arc. The centroid coincides, of course, with the centre, when such a centre exists and the distribution of mass is uniform.

17. As soon as ρ is given either as a constant or as a function of the co-ordinates, the problem of determining the centroid of a continuous mass is merely a problem in integration. To simplify the integrations, it is of importance to select the element in a convenient way conformably to the nature of the particular problem.

Considerations of symmetry and other geometrical properties will frequently make it possible to determine the centroid without resorting to integration. Thus, in a homogeneous mass, any plane of symmetry, or any axis of symmetry, must contain the centroid, since for such a plane or line the sum of the moments is evidently zero (see Art. 47).

It is to be observed that the whole discussion is entirely independent of the physical nature of the masses m which appear here simply as numerical coefficients, or "weights," attached to the points (comp. Art. 5). Some of the masses might even be negative.

It will be shown later that the *centre of gravity*, as well as the *centre of inertia*, of a body coincides with its centroid.

18. The centroid can be defined without any reference to a co-ordinate system as follows.

As in Art. 12, let there be given a system of n points $P_1, P_2, \dots, P_n$ (Fig. 1) whose masses are $m_1, m_2, \dots, m_n$. Taking an arbitrary origin O and putting $OP_1=r_1, OP_2=r_2, \dots, OP_n=r_n$, we may represent the moments $m_1r_1, m_2r_2, \dots, m_nr_n$ of the given masses with respect to O (Art. 11) by lengths (vectors) laid off on $OP_1, OP_2, \dots, OP_n$. The *moment of the system* can then be defined as the geometric sum of these vectors. It is therefore found by geometrically adding these vectors; *i.e.* we

have to lay off from O , on OP_1 , $Op_1 = m_1r_1$; from p_1 , parallel to OP_2 , $p_1p_2 = m_2r_2$, etc.; and finally join O to the end p_n of the polygon so formed; then Op_n is the geometric sum, or resultant, of the

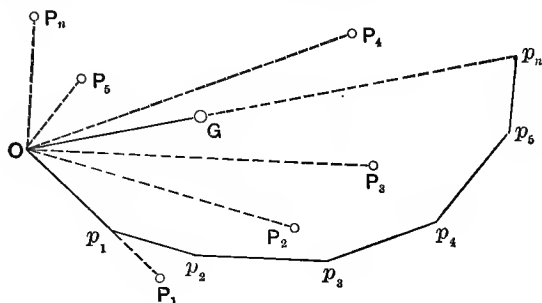


Fig. 1.

vectors $m_1r_1, m_2r_2, \dots, m_nr_n$. Using square brackets to indicate geometric addition, we have $Op_n = \Sigma[mr]$. A point G taken on the line Op_n so that

$$M \cdot OG = Op_n = \Sigma[mr], \quad (3)$$

where $M = \Sigma m$, is the *centroid of the system*.

19. It is easy to see that this definition of the centroid agrees with the one previously given (Art. 13). For, to form the geometric sum, or resultant, of the vectors $m_1r_1, m_2r_2, \dots, m_nr_n$, we may resolve each of these vectors along three rectangular axes drawn through O . The components of m_1r_1 are evidently m_1x_1, m_1y_1, m_1z_1 , if x_1, y_1, z_1 are the co-ordinates of P_1 , since $x_1/r_1, y_1/r_1, z_1/r_1$ are the direction cosines of the line OP_1 . We find therefore for the components of Op_n the values $\Sigma mx, \Sigma my, \Sigma mz$; and hence for the co-ordinates of G ,

$$\bar{x} = \Sigma mx / M, \quad \bar{y} = \Sigma my / M, \quad \bar{z} = \Sigma mz / M.$$

20. The position of the centroid G of a given system of masses is independent of the point O selected as origin. For let another point O' at the distance d from O be selected as

origin, and let G' be the point obtained as centroid from this origin, so that

$$M \cdot OG = \Sigma[mr], \quad M \cdot O'G' = \Sigma[mr'].$$

As we have the geometric equation $[r'] = [d] + [r]$, we find

$$M \cdot O'G' = \Sigma[md] + \Sigma[mr] = Md + \Sigma[mr].$$

Hence subtracting the first equation and dividing by M ,

$$[O'G'] - [OG] = [d], \text{ or } [O'G'] = [d] + [OG] = [O'G]$$

so that G and G' coincide.

It follows from this consideration that a given system has only one centroid.

21. Regarding again the mass of the centroid as equal to that of the whole system, we may now define the **centroid** of a system as *a point such that its moment with respect to any point or plane is equal to the sum of the moments of all the points constituting the system*; the sum being understood to be a geometric sum for moments with respect to a point, and an algebraic sum for moments with respect to a plane.

Taking the centroid itself as origin, we have the proposition that *the geometric sum of the moments of a system with respect to the centroid is equal to zero*. It has been proved before (Art. 14) that *the algebraic sum of the moments of a system vanishes for any plane passing through the centroid*.

22. In determining the centroid of a given system it will often be found convenient to break the system up into a number of partial systems whose centroids are either known or can be found more readily. *The moment of the whole system is obviously equal to the sum of the moments of the partial systems*.

Thus let the given mass M be divided into k partial masses $M_1, M_2, \dots, M_k$, so that $M = M_1 + M_2 + \dots + M_k$; let $G, G_1, G_2, \dots$

G_k be the centroids of $M, M_1, M_2, \dots M_k$, and $\bar{p}, \bar{p}_1, \bar{p}_2, \dots \bar{p}_k$ their distances from some fixed plane. Then we have

$$M\bar{p} = M_1\bar{p}_1 + M_2\bar{p}_2 + \dots + M_k\bar{p}_k.$$

23. The particular case of *two* partial systems occurs most frequently. We then have with reference to any plane

$$M\bar{p} = M_1\bar{p}_1 + M_2\bar{p}_2, \quad M = M_1 + M_2.$$

Letting the plane coincide successively with the three co-ordinate planes, it will be seen that G must lie on the line joining G_1, G_2 . Now taking the plane at right angles to $G_1 G_2$ through G_1 , we have

$$M \cdot G_1 G = M_2 \cdot G_1 G_2;$$

similarly for a plane through G_2 ,

$$M \cdot G G_2 = M_1 \cdot G_1 G_2;$$

whence
$$\frac{G_1 G}{M_2} = \frac{G G_2}{M_1} = \frac{G_1 G_2}{M};$$

i.e. the centroid of the whole system divides the distance of the centroids of the two partial systems in the inverse ratio of their masses.

3. EXAMPLES OF THE DETERMINATION OF CENTROIDS.

24. Two Particles. The centroid G of two particles of masses m_1, m_2 concentrated at two points P_1, P_2 lies on the line $P_1 P_2$ and divides the distance $P_1 P_2$ in the inverse ratio of their masses, *i.e.* so that

$$\frac{P_1 G}{m_2} = \frac{G P_2}{m_1} = \frac{P_1 P_2}{m_1 + m_2}.$$

(See Art. 23.) These formulæ hold even when one of the masses is positive and the other negative, in which case the sense of the segments must be attended to.

25. Three Particles. We find first the centroid P' of m_2 at P_2 and m_3 at P_3 (Fig. 2) by Art. 24; then, by the same rule, the centroid G of $m_2 + m_3$ at P' and m_1 at P_1 . We might have begun with P_3 and P_1 , finding P'' ; or with P_1 and P_2 , finding P''' . G lies at the intersection of the three lines P_1P' , P_2P'' , P_3P''' , and can therefore be constructed graphically.

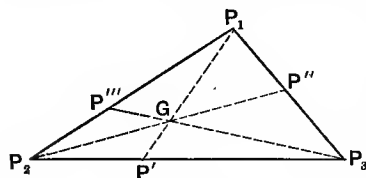


Fig. 2.

26. Four Particles. Find the centroid P' of m_1 at P_1 and m_2 at P_2 ; also the centroid P'' of m_3 at P_3 and m_4 at P_4 ; then the centroid G of $m_1 + m_2$ at P' and $m_3 + m_4$ at P'' .

The four particles can be arranged in groups of two in three different ways. There are therefore three lines, like $P'P''$, on each of which G lies. Any two of these are sufficient to construct G geometrically.

27. The centroid of a **homogeneous rectilinear segment** (thin rod or wire of constant cross-section) is evidently at its middle point.

28. If the density of a rectilinear segment be proportional to the n th power of the distance from one end, say $\rho = kx^n$, we have

$$\bar{x} = \frac{\int_0^l \rho x dx}{\int_0^l \rho dx} = \frac{k \int_0^l x^{n+1} dx}{k \int_0^l x^n dx} = \frac{n+1}{n+2} l,$$

where l is the length of the segment.

(a) For $n=0$, this gives $\bar{x} = \frac{1}{2}l$ which determines the centroid of a *homogeneous straight segment* (see Art. 27).

(b) For $n=1$, we have $\bar{x} = \frac{2}{3}l$. This determines the distance, from the vertex, of the centroid of a *homogeneous triangular area*. For such an area can be resolved (Fig. 3) by parallels to the base into elements each of which may be regarded as

a homogeneous segment PQ . If we imagine the mass of every such element concentrated at its middle point, the homogeneous triangle is replaced by its median CC' in which the density is proportional to the distance from the vertex C .

The centroid of a homogeneous triangular area lies therefore on the median at two-thirds of its length from the vertex; as this holds for each median, the intersection of the three medians is the centroid (see Art. 32).

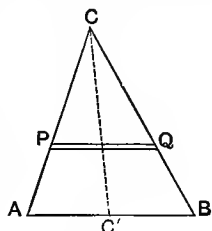


Fig. 3.

(c) For $n=2$, we have $\bar{x} = \frac{3}{4}l$. This gives the position of the centroid of a *homogeneous pyramid* or *cone*, by reasoning precisely similar to that used in (b).

Thus, to find the centroid of any homogeneous pyramid or cone, join the vertex to the centroid of the area of the base; the required centroid lies on this line at a distance equal to $\frac{3}{4}$ of its length from the vertex.

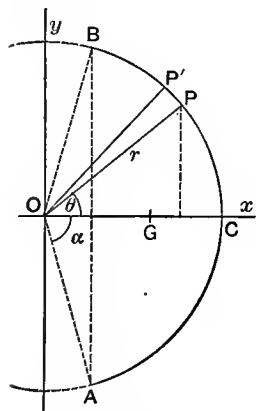


Fig. 4.

29. Homogeneous Circular Arc (Fig. 4).

Let O be the centre, r the radius of the circle; $ACB=s$ the arc, C its middle point. The centroid G must lie on the bisecting radius OC , since this being a line of symmetry, the sum of the moments of the elements of the arc is $=0$ with respect to this line (Art. 17). To find the distance $\bar{x}=OG$, we take moments with respect to the diameter perpendicular to OC . With OC as axis of x , we have

$$s \cdot \bar{x} = \int x ds = r \int \frac{x}{r} ds = r \int ds \cos COP = r \int dy.$$

Hence, $s \cdot \bar{x} = r \cdot c$, if c be the length of the chord AB .

If the angle $AOB=2\alpha$ of the arc AB were given, we might

obtain the result by taking the angle $COP = \theta$ as independent variable. We have then

$$s \cdot \bar{x} = \int_{-a}^{+a} r \cos \theta \cdot r d\theta = 2r^2 \sin \alpha,$$

whence
$$\bar{x} = r \cdot \frac{\sin \alpha}{\alpha}.$$

This can be written $\bar{x} = r \cdot \frac{2r \sin \alpha}{2r\alpha} = r \cdot \frac{c}{s}$, which agrees with the expression found above.

30. The First Proposition of Pappus and Guldinus. If an arc of a plane curve be made to rotate about an axis situated in its plane, it generates a surface of revolution whose surface-area is $S = 2\pi \int y ds$, where ds is the element of the curve and the axis of rotation is taken as axis of x . On the other hand we have, if s be the length of the generating arc and $\bar{y}$ the ordinate of its centroid, $s \cdot \bar{y} = \int y ds$; hence

$$S = 2\pi \cdot s\bar{y} = 2\pi \bar{y} \cdot s,$$

i.e. the surface-area of a solid of revolution is obtained by multiplying the generating arc into the path described by its centroid.

It is easy to see that this proposition holds even for incomplete revolutions. When the generating arc cuts the axis, proper regard must be had for signs and sense of rotation.

31. It follows from symmetry that the centroid of a **homogeneous circular** or **elliptic area** (plate, lamina) is at the geometrical centre of figure. Similarly, the centroid of a **homogeneous parallelogram** is at the intersection of its diagonals.

In general, if a homogeneous plane figure have two axes of symmetry, the centroid must be at the intersection of these lines since the sum of the moments is zero for each of these lines.

32. It has been shown in Art. 28 (*b*) how the centroid of a **homogeneous triangular area** ABC can be found.

Dividing the area into linear elements by drawing lines parallel to one of the sides, say AB (Fig. 3, p. 13), it appears that the centroid of each element, such as PQ , lies at its middle point. The locus of these middle points is the median CC' of the triangle; on this line, then, the centroid G of the triangle must be situated. Resolving the triangle into linear elements parallel to the side BC , or to CA , it follows in the same way that G must lie on each of the other two medians of the triangle. The intersection of these medians is therefore the centroid G .

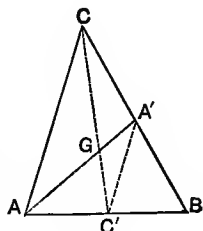


Fig. 5.

The point G trisects each median so that $CG/GC' = 2$. For if AA' (Fig. 5) is another median, the triangles AGC and $A'GC'$ are similar, and $A'C' = \frac{1}{2}AC$; hence $C'G = \frac{1}{2}CG$.

It follows from Art. 25, that the centroid of the homogeneous triangular area coincides with that of three particles of equal mass placed at the vertices.

33. Homogeneous Quadrilateral. The centroid is found graphically by resolving the quadrilateral into triangles, finding their centroids, and deducing from them the centroid of the quadrilateral. This process applies generally to *any polygon* and can be carried out in various ways.

Thus for the quadrilateral $ABCD$ (Fig. 6) drawing the diagonal AC and determining the centroids of the triangles

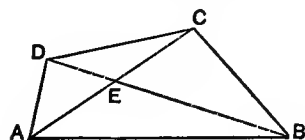


Fig. 6.

ABC and ADC , we obtain by joining these centroids one line on which the required centroid of the quadrilateral must lie. Repeating the same construction for the triangles obtained by drawing the

other diagonal BD , we find a second line on which the centroid

must lie. The intersection of these lines gives the centroid of the quadrilateral.

34. For some purposes it is convenient to find a system of particles whose centroid shall be the same as that of a quadrilateral. The problem is of course indeterminate and may be solved in various ways.

Let m be the mass of the quadrilateral $ABCD$; m_1, m_2 the masses of the triangles ABC, ADC . By Art. 32, each of these triangles can be replaced by three equal particles $\frac{1}{3}m_1, \frac{1}{3}m_2$, placed at the vertices. We thus have at A , as well as at C , a mass $\frac{1}{3}(m_1 + m_2) = \frac{1}{3}m$.

The masses $\frac{1}{3}m_1$ at B and $\frac{1}{3}m_2$ at D , whose sum is also $= \frac{1}{3}m$, are proportional to the areas of the triangles ABC, ADC , or to the lengths EB, ED , if E be the intersection of the diagonals. Now these two different masses at B and D can be replaced by a system of three masses, $\frac{1}{3}m$ at B , $\frac{1}{3}m$ at D , and $-\frac{1}{3}m$ at E . For (1) the total mass evidently remains the same, and (2) the centroids of the two systems coincide as is easily seen by taking moments with respect to E .

Indeed, the centroid G' of $\frac{1}{3}m_1$ at B and $\frac{1}{3}m_2$ at D is determined by the equation

$$(m_1 + m_2) \cdot EG' = m_1 \cdot EB - m_2 \cdot ED;$$

substituting for m_1, m_2 their values as found from the relations $m_1 + m_2 = m$, $m_1/m_2 = EB/ED$, this reduces to

$$m \cdot EG' = m \cdot (EB - ED).$$

The centroid G'' of $\frac{1}{3}m$ at B , $\frac{1}{3}m$ at D , and $-\frac{1}{3}m$ at E is given by

$$m \cdot EG'' = m \cdot EB - m \cdot ED - m \cdot 0.$$

Hence G' and G'' coincide.

The centroid of the area of a homogeneous quadrilateral is therefore the same as that of four equal particles placed at its

vertices together with a fifth particle of equal but negative mass, placed at the intersection of the diagonals.

35. In the particular case of a **homogeneous trapezoid** (Fig. 7), it may be noticed that the figure can be divided into rectilinear elements by lines drawn parallel to the parallel sides of the trapezoid. Every such element has its centroid at its middle point; the locus of all these points is the so-called median; and the centroid G of the trapezoid must lie on this median, *i.e.* on the line joining the middle points E, F of the parallel sides.

To find the ratio in which G divides the length EF , we use again the method of taking moments. We divide the trapezoid

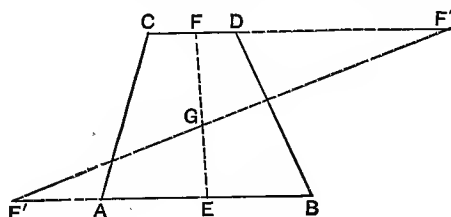


Fig. 7.

into two triangles by the diagonal BC and remember that the distance of the centroid of a triangle from its base is equal to one-third of its height; then taking moments with respect to the two parallel sides $AB=a$, $CD=b$, denoting the height of the trapezoid by h , and the distances of G from a and b by $\bar{y}$ and $\bar{y}'$, we obtain

$$\begin{aligned}\frac{1}{2}(a+b)h \cdot \bar{y} &= \frac{1}{2}ah \cdot \frac{1}{3}h + \frac{1}{2}bh \cdot \frac{2}{3}h, \\ \frac{1}{2}(a+b)h \cdot \bar{y}' &= \frac{1}{2}ah \cdot \frac{2}{3}h + \frac{1}{2}bh \cdot \frac{1}{3}h.\end{aligned}$$

Dividing, we find

$$\frac{\bar{y}}{\bar{y}'} = \frac{EG}{GF} = \frac{a+2b}{2a+b} = \frac{\frac{1}{2}a+b}{a+\frac{1}{2}b}.$$

This gives the following construction: Make $AE'=b$ on the prolongation of a , and $DF'=a$ on the prolongation of b , in the opposite sense; then $E'F'$ will intersect EF in G .

36. To find the centroid of the **cross-section of a T-iron** (Fig. 8) it is only necessary to find its distance $\bar{x}$ from the lower side AB ; for it must lie on the axis of symmetry CD . Taking moments with respect to AB we obtain with the notation indicated in the figure :

$$[2a\beta + 2\alpha(b-\beta)] \cdot \bar{x} = 2a\beta \cdot \frac{a}{2} + 2(b-\beta)\alpha \cdot \frac{a}{2},$$

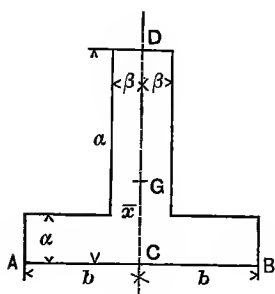


Fig. 8.

$$\text{hence } \bar{x} = \frac{1}{2} \frac{a^2\beta + b\alpha^2 - \alpha^2\beta}{a\beta + b\alpha - \alpha\beta}.$$

If α, β are nearly equal and very small in comparison with a, b , we have approximately

$$\bar{x} = \frac{1}{2} \frac{a^2 + b\alpha}{a + b}.$$

37. The area of a **homogeneous circular sector** (Fig. 4, p. 13) of radius r and angle $AOB = 2\alpha$ can be resolved into triangular elements $POP' = \frac{1}{2}r^2d\theta$, the bisecting radius OC being taken as polar axis. The centroid of such an element lies, by Art. 32, at the distance $\frac{2}{3}r$ from the centre O . Regarding the mass, $\rho \cdot \frac{1}{2}r^2d\theta$, of each element as concentrated at its centroid, the sector is replaced by a homogeneous circular arc of radius $\frac{2}{3}r$ and density $\frac{1}{2}\rho r^2d\theta$. By Art. 29, the centroid of such an arc, which is the required centroid of the sector, lies on the bisecting radius OC at the distance $\frac{2}{3}r \cdot \frac{\sin \alpha}{\alpha}$ from the centre O . Hence

$$\bar{x} = \frac{2}{3}r \frac{\sin \alpha}{\alpha}.$$

38. In general, for **areas bounded by curves** we must resort to integration, using the general formulæ of Art. 15.

If the area S be plane, we have in rectangular co-ordinates

$$M = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \rho dx dy,$$

$$M \cdot \bar{x} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \rho x dx dy, \quad M \cdot \bar{y} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \rho y dx dy;$$

and if the mass be homogeneous, *i.e.* $\rho = \text{const.}$, since then the first integration can at once be effected :

$$S = \int_{x_1}^{x_2} (y_2 - y_1) dx,$$

$$S \cdot \bar{x} = \int_{x_1}^{x_2} x (y_2 - y_1) dx, \quad S \cdot \bar{y} = \frac{1}{2} \int_{x_1}^{x_2} (y_2^2 - y_1^2) dx,$$

or similar expressions for y as independent variable.

In polar co-ordinates, the element of area is $r dr d\theta$, and we have $x = r \cos \theta$, $y = r \sin \theta$; hence

$$S = \iint r dr d\theta,$$

$$S \cdot \bar{x} = \iint r^2 \cos \theta dr d\theta, \quad S \cdot \bar{y} = \iint r^2 \sin \theta dr d\theta;$$

or, performing the first integration,

$$S = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 d\theta,$$

$$S \cdot \bar{x} = \frac{1}{3} \int_{\theta_1}^{\theta_2} r^3 \cos \theta d\theta, \quad S \cdot \bar{y} = \frac{1}{3} \int_{\theta_1}^{\theta_2} r^3 \sin \theta d\theta.$$

It will be noticed that these last formulæ express also that the infinitesimal sector $\frac{1}{2} r^2 d\theta$ is taken as element, the centroid of this element having the co-ordinates $\frac{2}{3} r \cos \theta$, $\frac{2}{3} r \sin \theta$.

39. As a somewhat more complicated example let us consider a circular disc of radius a , in which the density varies directly as the distance from the centre (Fig. 9). Let a circle described upon a radius as diameter be cut out of this disc; it is required to find the centroid of the remainder.

Let O be the centre of the disc of radius a , C that of the disc of radius $\frac{1}{2}a$; G_1 the centroid of the latter, G the required centroid; and put $OG_1 = \bar{x}_1$, $OG = \bar{x}$. Then if M_1 be the mass

of the smaller disc, M_2 that of the larger, we must have $(M_2 - M_1) \cdot \bar{x} = M_1 \bar{x}_1$.

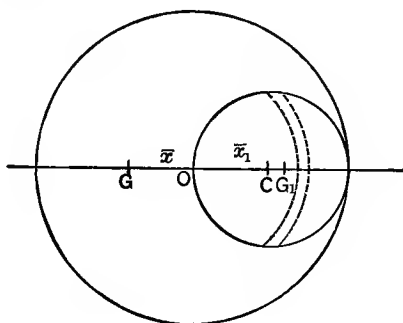


Fig. 9.

The equation of the smaller circle is $r = a \cos \theta$. Taking as element of the mass of the smaller disc the mass contained between two arcs of radii r and $r + dr$, we have for this element:

$$dM_1 = \rho \cdot 2 \theta r dr,$$

or since $\rho = kr$, $r = a \cos \theta$,

$$dM_1 = 2 ka^3 \theta \cos^2 \theta d(\cos \theta).$$

$$\begin{aligned} \text{Hence } M_1 &= \frac{2}{3} ka^3 \int_{\frac{\pi}{2}}^0 \theta d(\cos^3 \theta) \\ &= \frac{2}{3} ka^3 \left(\theta \cos^3 \theta - \int \cos^3 \theta d\theta \right)_{\frac{\pi}{2}}^0 \\ &= \frac{2}{3} ka^3 \int_0^{\frac{\pi}{2}} \cos^3 \theta d\theta = \frac{2}{3} ka^3 \cdot \frac{2}{3} = \frac{4}{9} ka^3. \end{aligned}$$

The centroid of the element dM_1 lies, according to Art. 29, at the distance $r \frac{\sin \theta}{\theta}$ from O . We have therefore

$$\begin{aligned} M_1 \bar{x}_1 &= -2 ka^3 \int_{\frac{\pi}{2}}^0 \theta \sin \theta \cos^2 \theta d\theta \cdot r \frac{\sin \theta}{\theta} \\ &= 2 ka^4 \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^3 \theta d\theta = \frac{4}{15} ka^4. \end{aligned}$$

The mass of the larger disc is

$$M_2 = \int_0^a k r \cdot 2 \pi r \cdot dr = 2 \pi k \int_0^a r^2 dr = \frac{2}{3} \pi k a^3.$$

Substituting these values into the equation of moments we find

$$\bar{x} = \frac{M_1 x_1}{M_2 - M_1} = \frac{6}{5(3\pi - 2)} a = 0.1616..a.$$

40. Proceeding to the determination of the centroids of *curved surface-areas*, we begin with the special case of the homogeneous area of a **surface of revolution**. If the axis of x coincide with the axis of revolution and $R = r \sin \theta$ be the distance of any point P of the surface from this axis (Fig. 10), the equation of the surface, or of its meridian section, is $x = f(R)$; and the element of area is

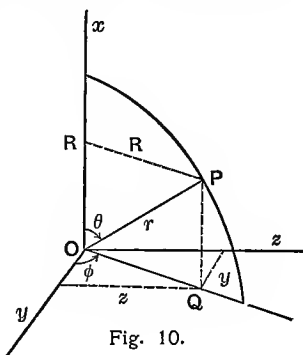


Fig. 10.

$$dS = R d\phi \sqrt{dx^2 + dR^2} = R \sqrt{1 + [f'(R)]^2} dR d\phi.$$

We have therefore for the centroid of the portion of the surface contained between two sections at right angles to the axis and two meridian planes (*i.e.* planes through the axis) including an angle ϕ :

$$S = \int_{R_1}^{R_2} \int_0^\phi R \sqrt{1 + f'^2} dR d\phi = \phi \int_{R_1}^{R_2} R^2 \sqrt{1 + f'^2} dR,$$

$$S \cdot \bar{x} = \int_{R_1}^{R_2} \int_0^\phi R f(R) \sqrt{1 + f'^2} dR d\phi = \phi \int_{R_1}^{R_2} R f(R) \sqrt{1 + f'^2} dR,$$

$$S \cdot \bar{y} = \int_{R_1}^{R_2} \int_0^\phi R^2 \cos \phi \sqrt{1 + f'^2} dR d\phi = \sin \phi \int_{R_1}^{R_2} R^2 \sqrt{1 + f'^2} dR,$$

$$S \cdot \bar{z} = \int_{R_1}^{R_2} \int_0^\phi R^2 \sin \phi \sqrt{1 + f'^2} dR d\phi = (1 - \cos \phi) \int_{R_1}^{R_2} R^2 \sqrt{1 + f'^2} dR.$$

Similar formulæ result when x is taken as independent variable instead of R . For a complete surface of revolution $\phi = 2\pi$ so that $\bar{y} = 0$, $\bar{z} = 0$, as is otherwise evident.

41. In the case of *spherical surfaces*, although the preceding formulæ can of course be used, it is often more convenient to make use of the geometrical property of the sphere that any spherical area is equal to the area of its projection on a cylinder circumscribed about the sphere.

Thus the *area on the sphere contained between two parallel planes* is equal to the area cut out by the same two planes from the circumscribed cylinder whose axis is perpendicular to the planes. The centroid of such a spherical area is therefore on the radius at right angles to the bounding planes midway between these planes.

42. **The Second Proposition of Pappus and Guldinus** (compare Art. 30).

A plane area S (Fig. 11) rotating about any axis situated in its plane generates a solid of revolution whose volume is $V = \pi \int (y_2^2 - y_1^2) dx$, if the axis of revolution is taken as axis of x and y_1, y_2 are the two ordinates of the curve bounding the area. On the other hand, if $\bar{y}$

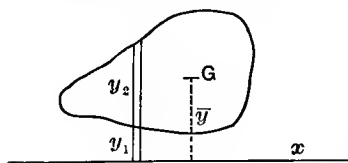


Fig. 11.

be the distance of the centroid G of the plane area from the axis, we have

$$S \cdot \bar{y} = \frac{1}{2} \int (y_2^2 - y_1^2) dx,$$

by Art. 38. Combining these two results, we find

$$V = 2\pi \bar{y} \cdot S,$$

i.e. *the volume of a solid of revolution is obtained by multiplying the generating area into the path described by its centroid.*

The proposition evidently holds even for a partial revolution.

43. To find the centroid of a **portion of any curved surface** $F(x, y, z)=0$, we have only to substitute $dM=\rho dS$ in the general formulæ of Art. 15, and then express dS by the ordinary methods of analytic geometry.

Denoting by l, m, n the direction cosines of the normal to the surface at the point (x, y, z) , and putting for shortness $\partial F/\partial x=F_x, \partial F/\partial y=F_y, \partial F/\partial z=F_z$, we have

$$dS = \frac{dydz}{l} = \frac{dzdx}{m} = \frac{dxdy}{n},$$

$$\frac{l}{F_x} = \frac{m}{F_y} = \frac{n}{F_z} = \frac{1}{\sqrt{F_x^2 + F_y^2 + F_z^2}}.$$

Hence, substituting

$$dS = dxdy \cdot \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{F_z}$$

in the formulæ of Art. 15, we find

$$M = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \rho dxdy \cdot \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{F_z},$$

where the integration is to be extended over the projection of the portion of surface under consideration on the plane xy . The equation of the curve bounding this projection must be given: it determines the limits of integration. It is obvious how the formula has to be modified when the projection of the area on either of the other co-ordinate planes be given.

The expressions for $M \cdot \bar{x}$, $M \cdot \bar{y}$, $M \cdot \bar{z}$ differ from the above expression for M only in containing the additional factor x, y, z , respectively, under the integral sign.

44. If the equation of the surface be given in the form $z=f(x, y)$, as is frequently the case, we have

$$F(x, y, z) \equiv z - f(x, y);$$

hence with the usual Gaussian notation

$$\frac{\partial z}{\partial x} \equiv \frac{\partial f}{\partial x} \equiv p, \quad \frac{\partial z}{\partial y} \equiv \frac{\partial f}{\partial y} \equiv q,$$

$$F_x = -p, \quad F_y = -q, \quad F_z = 1,$$

which gives

$$M = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \rho \sqrt{1 + p^2 + q^2} \, dx dy,$$

$$M \cdot \bar{x} = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \rho x \sqrt{1 + p^2 + q^2} \, dx dy,$$

$$M \cdot \bar{y} = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \rho y \sqrt{1 + p^2 + q^2} \, dx dy,$$

$$M \cdot \bar{z} = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \rho z \sqrt{1 + p^2 + q^2} \, dx dy.$$

In the case of a homogeneous *spherical surface* $x^2 + y^2 + z^2 = a^2$, we have $p = \partial z / \partial x = -x/z$, $q = \partial z / \partial y = -y/z$; hence $z \sqrt{1 + p^2 + q^2} = a$, so that the last of the above formulæ gives

$$S \cdot \bar{z} = a \iint dx dy = a \cdot S_z,$$

where S is the area of the surface and S_z the area of its projection on the plane xy . The formula shows that the distance $\bar{z}$ of the centroid of any spherical area S from a plane passing through the centre is equal to the radius a multiplied by the ratio of the projection S_z of the area on the plane to the area itself.

45. We proceed to the methods of finding the centroids of volumes or solids.

Considerations of symmetry make it clear that the centroid of a **homogeneous parallelepiped** lies at the intersection of its diagonals; similarly, that of a **homogeneous prism** or **cylinder** coincides with the centroid of the area of its middle section (*i.e.* a plane section parallel to, and equally distant from, the bases).

46. For a **homogeneous pyramid** or **cone**, we have found in Art. 28 (c) that the centroid lies on the line joining the vertex to the centroid of the area of the base, at a distance equal to $\frac{1}{4}$ of this line from the base. This is, of course, easily shown directly by resolving the pyramid or cone into plane elements parallel to the base, in a manner analogous to that used for the triangular area in Art. 32.

47. It may, perhaps, be well to formally state the principal **laws of symmetry** for homogeneous solids, although they present themselves so naturally that they are used almost instinctively. For however simple and obvious these propositions may appear, the beginner may be led into error if he does not use them cautiously. The proof rests on the fundamental definition of the centroid as a point such that for any plane through it the sum of the moments is zero.

(a) *If the surface of the solid have a plane of symmetry, i.e. a plane such that every line perpendicular to it intersects the surface in two points equidistant from the plane, the centroid lies in this plane.* Hence, the centroid of a homogeneous solid is at once known if its surface possesses three planes of symmetry. If the surface has two planes of symmetry, the centroid lies on their line of intersection.

(b) *If the surface have an axis of symmetry, i.e. a line such that every line perpendicular to it intersects the surface in two points equidistant from the line, the centroid must lie on this axis.* Two axes of symmetry in the same homogeneous solid determine its centroid by their intersection.

(c) *If the surface have a centre, i.e. a point such that every line through it intersects the surface in two points equidistant from it, the centroid coincides with this centre.*

(d) *If the surface have a diametral plane, i.e. a plane bisecting all chords that are parallel to a certain direction, the centroid lies in this plane.*

48. Homogeneous spherical solids can be treated by a method analogous to that used for circular areas (see Art. 37). Thus a **homogeneous spherical sector** can be resolved into infinitesimal elements, each of which is a pyramid whose vertex lies at the

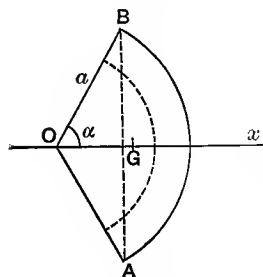


Fig. 12.

centre of the sphere and whose base is an infinitesimal element of the spherical surface area of the sector. Such an element, regarded as a pyramid (Art. 46), has its centroid at the distance $\frac{3}{4}a$ from the centre, if a be the radius of the sphere. We may regard its mass as concentrated at its centroid and have thus the solid sector replaced by a homogeneous segment of a spherical area, of radius $\frac{3}{4}a$. It has been shown in Art. 41 that the centroid of such a segment bisects its height.

Let 2α be the angle at the vertex of the given sector (Fig. 12); then the height of the segment of radius $\frac{3}{4}a$ is $\frac{3}{4}a(1 - \cos \alpha)$; hence the distance $\bar{x}$ of the centroid of the solid spherical sector from the centre is

$$\bar{x} = \frac{3}{4}a \cos \alpha + \frac{3}{8}a(1 - \cos \alpha) = \frac{3}{8}a(1 + \cos \alpha) = \frac{3}{4}a \cos^2 \frac{\alpha}{2}.$$

49. In a homogeneous solid of revolution the centroid lies on the axis of revolution, since this line is an axis of symmetry (Art. 47 (b)). Taking this line as the axis of x , the equation of the surface of the solid is determined by that of the curve bounding the generating area, say $y=f(x)$.

We select as element the circular or ring-shaped plate of thickness dx contained between two sections of the solid at right angles to the axis of revolution (Fig. 11, p. 22). The centroid of each such element lies on the axis, and the volume of the element is $\pi(y_2^2 - y_1^2)dx$, if y_1, y_2 , are the ordinates of the curve corresponding to the same value of x .

We have, therefore,

$$\bar{x} = \frac{\int_{x_1}^{x_2} x (y_2^2 - y_1^2) dx}{\int_{x_1}^{x_2} (y_2^2 - y_1^2) dx}, \quad \bar{y} = 0, \quad \bar{z} = 0.$$

It is easy to see how the formula has to be modified when only one value or more than two values of y correspond to a given value of x .

50. In the most general case of **any solid** whatever the formulæ of Art. 15 assume different forms according to the system of co-ordinates used. Thus for rectangular Cartesian co-ordinates the element of volume is $dv = dx dy dz$, and we have :

$$M = \iiint \rho dx dy dz, \quad M \cdot \bar{x} = \iiint \rho x dx dy dz,$$

$$M \cdot \bar{y} = \iiint \rho y dx dy dz, \quad M \cdot \bar{z} = \iiint \rho z dx dy dz.$$

51. In polar co-ordinates, *i.e.* for the radius vector r , the co-latitude θ and the longitude ϕ (Fig. 10, p. 21), the element of volume is an infinitesimal rectangular parallelepiped having the concurrent edges dr , $r d\theta$, $r \sin \theta d\phi$; hence

$$dv = r^2 \sin \theta dr d\theta d\phi.$$

As $x = r \cos \theta$, $y = r \sin \theta \cos \phi$, $z = r \sin \theta \sin \phi$, the centroid is determined by the equations :

$$M = \iiint \rho r^2 \sin \theta dr d\theta d\phi,$$

$$M \cdot \bar{x} = \iiint \rho r^3 \sin \theta \cos \theta dr d\theta d\phi,$$

$$M \cdot \bar{y} = \iiint \rho r^3 \sin^2 \theta \cos \phi dr d\theta d\phi,$$

$$M \cdot \bar{z} = \iiint \rho r^3 \sin^2 \theta \sin \phi dr d\theta d\phi.$$

52. As an illustration let us determine the centroid of the

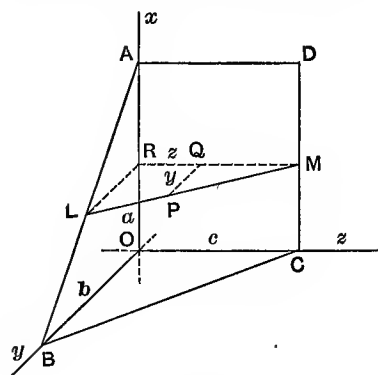


Fig. 13.

volume $OABCD$ (Fig. 13), bounded by the three co-ordinate planes and the warped quadrilateral (hyperbolic paraboloid) $ABCD$. The latter is generated by the line LM gliding along AB and CD so as to remain parallel to the plane yz . The data are $OA = CD = a$, $OB = b$, $OC = AD = c$.

We take as element an infinitesimal prism PQ of base $dxdz$ and height y . From similar triangles we have $y/RL = (c-z)/c$, and $RL/b = (a-x)/a$; hence

$$y = b \frac{a-x}{a} \cdot \frac{c-z}{c}.$$

Thus we find, rejecting the constants which cancel in numerator and denominator,

$$\begin{aligned} \bar{x} &= \frac{\int_0^a \int_0^c x(a-x)(c-z) dx dz}{\int_0^a \int_0^c (a-x)(c-z) dx dz} = \frac{\int_0^a x(a-x) dx \cdot \left(c^2 - \frac{c^2}{2}\right)}{\int_0^a (a-x) dx \cdot \left(c^2 - \frac{c^2}{2}\right)} \\ &= \frac{\int_0^a x(a-x) dx}{\int_0^a (a-x) dx} = \frac{\frac{a^3}{2} - \frac{a^3}{3}}{a^2 - \frac{a^2}{2}} = \frac{\frac{1}{6}a^3}{\frac{1}{2}a^2} = \frac{1}{3}a; \\ \bar{y} &= \frac{\frac{1}{2} \int_0^a \int_0^c b^2 \frac{(a-x)^2}{a^2} \cdot \frac{(c-z)^2}{c^2} dx dz}{\int_0^a \int_0^c b \frac{a-x}{a} \cdot \frac{c-z}{c} dx dz} = \frac{b \int_0^a (a-x)^2 dx \cdot \frac{1}{3}c^3}{2ac \int_0^a (a-x) dx \cdot \frac{1}{2}c^2} \\ &= \frac{1}{3} \cdot \frac{b}{a} \cdot \frac{3}{2} = \frac{2}{3}b. \end{aligned}$$

Finally, $\bar{z} = \frac{1}{3}c$, by analogy with $\bar{x}$.

53. Exercises.

(1) Three beads of masses 3, 5, 12, are strung on a straight wire whose mass is neglected, the bead of mass 5 being midway between the other two. Find the centroid. (Take moments about the middle point.)

(2) Show that the centroid of three equal particles placed at the vertices of a triangle is at the intersection of the medians of the triangle.

(3) Show that the centroid of three masses m_1, m_2, m_3 situated at the vertices of a triangle and proportional to the opposite sides, is at the centre of the inscribed circle.

(4) Equal particles are placed at five of the six vertices of a regular hexagon. Find the distance of the centroid from the centre of figure.

(5) Find the centroid of a homogeneous triangular frame.

(6) Show that the centroid of a homogeneous semicircular wire lies at the distance $\frac{2}{\pi}r$ from the centre, r being the radius.

(7) Find the co-ordinates of the centroid of the arc of a quadrant of a circle by using the first proposition of Pappus (Art. 30).

(8) Find the centroid of a circular arc AB of angle $AOB = \alpha$, whose density varies as the length of the arc measured from A .

Find the centroids of the following homogeneous arcs of curves :

(9) Parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum.

(10) Cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, from cusp to cusp.

(11) Half the cardioid $r = a(1 + \cos \theta)$.

(12) Catenary $y = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$ between two points equally distant from the axis of x .

(13) Common helix: $x = r \cos \theta$, $y = r \sin \theta$, $z = kr\theta$, from $\theta = 0$ to $\theta = \theta$.

(14) The sides of a right-angled triangle are a and b . Find the distances of the centroid of the triangular area from the vertices.

(15) From a square $ABCD$ one corner EAF is cut off so that $AE = \frac{3}{4}a$, $AF = \frac{1}{4}a$, a being the side of the square. Find the centroid of the remaining area.

(16) In a trapezoid the parallel sides are a, b , the height is h , and one of the non-parallel sides is perpendicular to the parallel sides; show that the co-ordinates of the centroid with a as axis of x and the perpendicular side as axis of y are $\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)}$, $\bar{y} = \frac{(a+2b) \cdot h}{3(a+b)}$.

(17) Find the centroid of the cross-section of a bar formed by placing four angle-irons with their edges together, two of the irons having the dimensions a, b, α, β , as in Fig. 8, Art. 36, while the other two have the dimension a different, say a' .

(18) Find the centroid of the cross-section of a U-iron, the length of the flanges being $a = 12$ in., that of the web $2b = 8$ in., and the thickness $\delta = 1$ in. Deduce the general formula for $\bar{x}$, and an approximate formula for a small δ , and compare the numerical results.

(19) In the cross-section of an unsymmetrical double T, the flanges are $2b = 12$ in., $2b' = 8$ in.; the web is $a = 10$ in.; and the thickness of each of the two channel-irons forming the bar is $\delta = 1$ in. throughout; find the centroid.

(20) In a T-iron the width of the flange is b , its thickness α ; the depth of the web is a , its thickness β . Find the distance of the centroid from the outer side of the flange; give an approximate expression and investigate it for $a = b$, $\alpha = \beta = \frac{1}{5}a$.

(21) If one-fourth be cut away from a triangle by a parallel to the base, show that in the remaining area the centroid divides the median in the ratio 4 : 5.

(22) Prove that the centroid of any plane quadrilateral $ABCD$ coincides with that of the triangle ACF , if the point F be constructed by laying off $BF = DE$ on the diagonal BD , E being the intersection of the diagonals.

(23) The centroid of a homogeneous semicircular area of radius r lies at the distance $\bar{x} = \frac{4}{3\pi}r$ from the centre.

(24) The centroid of the area of a homogeneous circular segment of radius r subtending at the centre an angle 2α is at the distance $\bar{x} = \frac{2}{3}r \cdot \frac{\sin^3 \alpha}{\alpha - \sin \alpha \cos \alpha}$, or, $\bar{x} = \frac{1}{6} \frac{c^3}{rs - ch}$, if c is the chord, h its distance from the centre, and s the arc.

(25) A painter's palette is formed by cutting a small circle of radius b out of a circular disc of radius a , the distance between the centres being c . It is required to find the distance of the centroid of the remainder from the centre of the larger circle. (Routh.)

(26) The arch constructed of brick over a door is in the form of a quadrant of a circular ring. The door is 5 ft. wide; $1\frac{1}{2}$ lengths of brick are used (say 12 in.). Find the centroid of the arch.

Find the co-ordinates of the centroid for the following plane areas :

(27) Area bounded by the parabola $y^2 = 4ax$, the axis of x , and the ordinate y .

(28) Area bounded by the curve $y = \sin x$ from $x = 0$ to $x = \pi$ and the axis of x .

(29) Quadrant of an ellipse.

(30) Elliptic segment bounded by the chord joining the ends of the major and minor axes.

(31) Show, by Art. 28, that the centroid of the surface of a right circular cone lies at a distance from the base equal to one-third of the height.

(32) Find the centroid of the portion of the surface of a right circular cone cut out by two planes through the axis inclined at an angle ϕ .

(33) Find the centroid of the area of the earth's surface contained between the tropic of Cancer (latitude = $23^\circ 28'$) and the arctic circle (polar distance = $23^\circ 28'$).

(34) Regarding the earth as a homogeneous sphere of density $\rho = 5.5$, how much would its centroid be displaced by superimposing over the area bounded by the arctic circle an ice-cap of a uniform thickness of 10 miles?

(35) A bowl in the form of a hemisphere is closed by a circular lid of a material whose density is three times that of the bowl. Find the centroid.

(36) Determine the centroid of a homogeneous solid hemisphere.

(37) Find the centroid of a frustum of a cone, the radii of the bases being r_1, r_2 ; the height of the frustum, h .

(38) Show that the formula for the frustum of the cone applies likewise to the frustum of any pyramid of the same height h if r_1, r_2 are any two homologous linear dimensions of the two bases.

(39) Find the centroid of a solid segment of a sphere of radius a , the height of the segment being h .

(40) Show that, both for a triangular area and for a tetrahedral volume, the distance of the centroid from any plane is the arithmetic mean of the distances of the vertices from the same plane.

(41) Find the centroid of the paraboloid of revolution of height h , generated by the complete revolution of the parabola $y^2 = 4ax$ about its axis.

(42) The area bounded by the parabola $y^2 = 4ax$, the axis of x , and the ordinate $y = y_1$, revolves about the tangent at the vertex. Find the centroid of the solid of revolution so generated.

(43) The same area as in problem (42) revolves about the ordinate y_1 . Find the centroid.

(44) Find the centroid of an octant of an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(45) The equations of the common cycloid referred to a cusp as origin and the base as axis of x are $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$. Find the centroid: (a) of the arc of the semi-cycloid (*i.e.* from cusp to vertex); (b) of the plane area included between the semi-cycloid and the base; (c) of the surface generated by the revolution of the semi-cycloid about the base; (d) of the volume generated in the same case; (e) of the surface generated by the revolution of the whole cycloid (from cusp to cusp) about its axis, *i.e.* the line through the vertex at right angles to the base; (f) of the volume so generated.

(46) Find the centroid of a solid hemisphere whose density varies as the n th power of the distance from the centre.

(47) From out of the right cone ABC a cone ABD is cut of the same base and axis, but of smaller height. Find the centroid of the remaining solid.

(48) A triangle ABC , whose sides are a, b, c , revolves about an axis situated in its plane. Find the surface area and volume of the solid so generated, if p, q, r are the distances of A, B, C from the axis.

(49) "Water is poured gently into a cylindrical cup of uniform thickness and density. Prove that the locus of the centre of gravity of the water, the cup, and its handle is a hyperbola." (Routh.)

(50) Prove that the volume of a truncated right cylinder (*i.e.* a right cylinder cut by a plane inclined at any angle to its base) is equal to the product of the area of its base into the height of the truncated cylinder at the centroid of its base.

(51) Prove that the volume of a doubly truncated cylinder is equal to the product of the area of the section at right angles to the axis into the distance of the centroids of the bases.

54. For the theory of moments and centres of mass the student is referred to W. SCHELL, *Theorie der Bewegung und der Kräfte*, Leipzig, Teubner, Vol. I., 1879, pp. 81-100; E. J. ROUTH, *Analytical statics*, Cambridge, University Press, Vol. I., 1891, pp. 270-314; J. SOMOFF, *Theoretische Mechanik*, übersetzt von A. Ziwet, Leipzig, Teubner, Vol. II., 1879, pp. 1-72. For problems see in particular W. WALTON, *Problems in illustration of the principles of theoretical mechanics*, Cambridge, Deighton, 1876, pp. 1-45; M. JULLIEN, *Problèmes de mécanique rationnelle*, Paris, Gauthier-Villars, Vol. I., 1866, pp. 1-46; F. KRAFT, *Probleme der analytischen Mechanik*, Stuttgart, Metzler, Vol. I., 1884, pp. 527-617. Compare, also, B. PRICE, *Infinitesimal calculus*, Oxford, Clarendon Press, Vol. III., 1868, pp. 163-206; MOIGNO, *Leçons de mécanique analytique, Statique*, Paris, Gauthier-Villars, 1868, pp. 106-206; G. MINCHIN, *Treatise on statics*, Oxford, Clarendon Press, Vol. I., 1884, pp. 261-305; I. TODHUNTER, *Analytical statics*, edited by J. D. Everett, London, Macmillan, 1887, pp. 115-189; W. WALTON, *Problems in elementary mechanics*, London, Bell, 1880, pp. 56-78; and for geometrical methods, the works on graphical statics.

II. *Momentum; Force; Energy.*

55. Let us consider a point moving uniformly from rest in a straight line. We know from *Kinematics* (Art. 111) that its motion is determined by the equations

$$v = jt, \quad s = \frac{1}{2}jt^2, \quad \frac{1}{2}v^2 = js, \quad (1)$$

where s is the distance passed over in the time t , v the velocity, and j the acceleration at the time t .

If, now, for the single point we substitute an m -tuple point, *i.e.* if we endow our point with the mass m , and thus make it a *particle* (see Art. 6), the equations (1) must be multiplied by m , and we obtain

$$mv = mjt, \quad ms = \frac{1}{2}mjt^2, \quad \frac{1}{2}mv^2 = mjs. \quad (2)$$

The quantities mv , mj , $\frac{1}{2}mv^2$ occurring in these equations have received special names because they correspond to certain physical conceptions of great importance.

56. The product mv of the mass m of a particle into its velocity v is called the **momentum**, or the **quantity of motion**, of the particle.

57. In observing the behaviour of a physical body in motion, we notice that the effect it produces—for instance, when impinging on another body, or more generally, whenever its velocity is changed—depends not only on its velocity, but also on its mass. Familiar examples are the following: a loaded railroad car is not so easily stopped as an empty one; the destructive effect of a cannon-ball depends both on its velocity and on its mass; the larger a fly-wheel, the more difficult is it to give it a certain velocity; etc.

It is from experiences of this kind that the physical idea of mass is derived.

The fact that any change of motion in a physical body is affected by its mass is sometimes ascribed to the so-called “*inertia*,” or “force of inertia,” of matter, which means, however, nothing else but the property of possessing mass.

58. Momentum, being by definition (Art. 56) the product of mass and velocity, has for its *dimensions* (see *Kinematics*, Art. 92)

$$\mathbf{MV} = \mathbf{MLT}^{-1}.$$

The *unit of momentum* is the momentum of the unit of mass having the unit of velocity.

Thus in the C.G.S. system the unit of momentum is the momentum of a particle of 1 gramme moving with a velocity of 1 cm. per second. There is no generally accepted name for this unit, although the name **bole** was proposed by the Committee of the British Association.

In the F.P.S. system, the unit is the momentum of a particle of one pound mass moving with a velocity of 1 ft. per second.

To find the relations between these two units, let there be x C.G.S. units in the F.P.S. unit; then

$$x \cdot \frac{\text{gm. cm.}}{\text{sec.}} = 1 \cdot \frac{\text{lb. ft.}}{\text{sec.}};$$

hence
$$x = \frac{\text{lb.}}{\text{gm.}} \cdot \frac{\text{ft.}}{\text{cm.}},$$

or, by Art. 3 and *Kinematics*, Art. 14,

$$x = 453.59 \times 30.48 = 13\,825.3;$$

i.e. 1 F.P.S. unit of momentum = 13 825.3 C.G.S. units, and
1 C.G.S. unit = 0.000 072 331 F.P.S. units.

59. Exercises.

(1) What is the momentum of a cannon-ball weighing 200 lbs. when moving with a velocity of 1500 ft. per second?

(2) With what velocity must a railroad-truck weighing 3 tons move to have the same momentum as the cannon-ball in Ex. (1)?

(3) Determine the momentum of a one-ton ram after falling through 20 feet.

60. The product mj of the mass m of a particle into its acceleration j is called **force**. Denoting it by F , we may write our equations (2) in the form

$$mv = Ft, \quad s = \frac{1}{2} \frac{F}{m} t^2, \quad \frac{1}{2} mv^2 = Fs. \quad (3)$$

As long as the velocity of a particle of constant mass remains constant, its momentum remains unchanged. If the velocity changes uniformly from the value v at the time t to v' at the time t' , the corresponding change of momentum is

$$mv' - mv = mjt' - mjt = F(t' - t); \quad (4)$$

hence
$$F = \frac{mv' - mv}{t' - t}. \quad (5)$$

Here the acceleration, and hence the force, was assumed constant. If F be variable, we have in the limit when $t' - t$ becomes dt ,

$$F = \frac{d(mv)}{dt} = m \frac{dv}{dt}. \quad (6)$$

Instead of defining force as the product of mass and acceleration, we may therefore define it as the *rate of change of momentum with the time*.

61. Integrating equation (6), we find

$$\int_t^{t'} F dt = mv' - mv. \quad (7)$$

The product $F(t' - t)$ of a constant force into the time $t' - t$ during which it acts, and in the case of a variable force, the time-integral $\int_t^{t'} F dt$, is called the **impulse** of the force during this time.

It appears from the equations (4) and (7) that *the impulse of a force during a given time is equal to the change of momentum during that time*.

62. The idea of force is no doubt primarily derived from the sensation produced in a person by the exertion of his "muscular force."

Like the sensations of light, sound, heat, etc., the sensation of exerting force is capable, in a rough way, of measurement. But the physiological and psychological phenomena attending the exertion of muscular force when analysed more carefully are very complicated.

In ordinary language the term "force" is applied in a great variety of meanings. For scientific purposes it is of course necessary to attach a single definite meaning to it.

63. In physics it is customary to speak of force as *producing or generating velocity*, and to define force as the *cause of acceleration*. Thus observation shows that the velocity of a falling body increases during the fall; the cause of the observed change in the velocity, *i.e.* of the acceleration, is called the force of attraction, and is supposed to be exerted by the earth. Again, a body falling in the air, or in some other medium, is observed to increase its velocity less rapidly than a body falling *in vacuo*; a force of resistance is therefore ascribed to the medium as the cause of this change. In a similar way we speak of the expansive force of steam, of electric and magnetic forces, etc., because all these agencies produce changes of velocity.

Now, any change in the velocity v of a body of given mass m implies a change in its momentum mv ; and it is this change of momentum, or rather the rate at which the momentum changes with the time, which is of prime importance in all the applications of mechanics. It is therefore convenient to have a special name for this rate of change, and that is what is called *force*.

It is, however, well to remember that in using this term "force," it is not intended to assert anything as to the objective reality or actual nature of force and matter in the ordinary acceptance of these terms. Our knowledge comes to us through our sense-impressions, and these would all seem to reduce finally to changes of motion and changes of momentum: these alone we can perceive directly.

64. The definition of force (Art. 60) as the product of mass and acceleration gives the *dimensions* of force as

$$\mathbf{F} = \mathbf{MJ} = \mathbf{MLT}^{-2}.$$

The *unit of force* is therefore the force of a particle of unit mass moving with unit acceleration.

Hence, in the C.G.S. system, it is the force of a particle of

1 gramme moving with an acceleration of 1 cm. per second per second. This unit force is called a **dyne**.

The definition is sometimes expressed in a slightly different form.* We may say the dyne is the force which, acting on a gramme uniformly for one second, would generate in it a velocity of 1 cm. per second; or would give it the C.G.S. unit of acceleration; or it is the force which, acting on *any* mass uniformly for one second, would produce in it the C.G.S. unit of momentum.

That these various statements mean the same thing follows from the fundamental formulæ $F = mj$, $j = vt$, if F , m , t , v , j be expressed in C.G.S. units.

65. In the F.P.S. system, the unit of force is the force of a mass of 1 lb. moving with an acceleration of 1 ft. per second per second. It is called the **poundal**.

66. The dyne and the poundal are called the **absolute**, or scientific, units of force.

To find the relation between these two units, let x be the number of dynes in the poundal; then we have

$$x \cdot \frac{\text{gm. cm.}}{\text{sec.}^2} = 1 \cdot \frac{\text{lb. ft.}}{\text{sec.}^2};$$

hence, just as in Art. 58,

$$x = 13\,825.3;$$

i.e. 1 poundal = 13 825.3 dynes, and 1 dyne = 0.000 072 331 poundals.

67. Another system of measuring force, the so-called **gravitation** (or engineering) system, is in very common use, and must here be explained.

Among the forces of nature the most common is the *force of gravity*, or the *weight*, *i.e.* the force with which any physical body is attracted by the earth. As we have convenient and

* J. D. EVERETT, *C.G.S. system of units*, 1891, p. 23, 24.

accurate appliances for comparing the weights of different bodies at the same place, the idea suggests itself of selecting as unit force the weight of a certain standard mass.

In the metric gravitation system the *weight of a kilogramme* has been selected as unit force; in the British gravitation system, the *weight of a pound* is the unit force.

68. There are two serious objections to the gravitation system of measuring force, one of a practical nature, the other theoretical. The former is that the words "kilogramme" and "pound" are thus used in two different meanings, sometimes, and more correctly, as denoting a mass, sometimes as denoting a force. Wherever an ambiguity might arise from this double use, the word "mass" or "weight" must be added.

The other objection is more serious. The weight of a body, and hence the gravitation unit of force, is not a constant quantity; it changes from place to place as it depends on the value of g , the acceleration of gravity.

For, the weight W of any mass m being the force with which this mass is attracted by the earth, we have

$$W = mg,$$

where g is the acceleration produced by the earth's attraction. Now it is known from experiment that this acceleration varies from place to place; according to the law of gravitation, it is inversely proportional to the square of the distance from the centre of the earth.

The weight of a body is therefore a meaningless term unless the place be specified where the body is situated, and the value of g at that place be given.

It is true, however, that the value of g for different points on the earth's surface varies but little, so that for most practical purposes the gravitation system is accurate enough.

In the equations of theoretical dynamics, in particular in kinetics, the use of absolute units is always understood. In statics, however, where we are mainly concerned with the *ratios* of forces and not with their absolute values, gravitation units will generally be used in the present work in view of the practical applications.

69. The numerical relation between the absolute and gravitation measures of force is expressed by the equations

1 kilogramme (force) = 1000 g dynes,

1 pound (force) = g poundals,

where g is about 981 in metric units, and about 32.2 in British units. In most cases the more convenient values 980 and 32 may be used.

70. Exercises.

(1) What is the exact meaning of "a force of 10 tons"? Express this force in poundals and in dynes.

(2) Reduce 2 000 000 dynes to British gravitation measure.

(3) Express a pressure of 2 lbs. per square inch in kilogrammes per square centimetre.

(4) Prove that a poundal is very nearly half an ounce, and a dyne a little over a milligramme, in gravitation measure.

(5) The numerical value of a force being 100 in (absolute) F.P.S. units, find its value for the yard as unit of length, the ton as unit of mass, and the minute as unit of time (see Art. 66).

71. *The quantity $\frac{1}{2}mv^2$, i.e. half the product of the mass of a particle into the square of its velocity, is called the kinetic energy of the particle.*

Let us consider again a particle of constant mass m moving with a constant acceleration, and hence with a constant force; let v be the velocity, s the space described at the time t ; v' , s' the corresponding values at the time t' . Then the last of the three fundamental equations (see Arts. 55 and 60) gives

$$\frac{1}{2}mv'^2 - \frac{1}{2}mv^2 = F(s' - s); \quad (8)$$

hence
$$F = \frac{\frac{1}{2}mv'^2 - \frac{1}{2}mv^2}{s' - s}. \quad (9)$$

If F be variable, we have in the limit

$$F = \frac{d(\frac{1}{2}mv^2)}{ds} = m \frac{dv}{ds}. \quad (10)$$

Force can therefore be defined as *the rate at which the kinetic energy changes with the space*. (Compare the end of Art. 60.)

72. Integrating the last equation (10), we find

$$\int_s^{s'} F ds = \frac{1}{2} m v'^2 - \frac{1}{2} m v^2. \quad (11)$$

*The product $F (s' - s)$ of a constant force F into the space $s' - s$ described in the direction of the force, and in the case of a variable force, the space-integral $\int_s^{s'} F ds$, is called the **work** of the force for this space.*

The equations (8) and (11) show that *the work of a force is equal to the corresponding change of the kinetic energy*.

We have here assumed that the force acts in the direction of motion of the particle. A more general definition of work including the above as a special case will be given later (Art. 230 sq.).

The ideas of energy and work have attained the highest importance in mechanics and mathematical physics within comparatively recent times. Their full discussion belongs to Kinetics.

73. According to their definitions, both *momentum* (Art. 56) and *force* (Art. 60) may be regarded mathematically as mere numerical multiples of velocity and acceleration, respectively. They are therefore so-called vector-quantities; *i.e.* a momentum as well as a force can be represented geometrically by a segment of a straight line of definite length, direction, and sense. Moreover, as they are referred to a particular point, *viz.* to the point whose mass is m , the line representing a momentum or a force must be drawn through this point; the line has therefore not only direction, but also position; *i.e.* a momentum as well as a force is represented geometrically by a **rotor** (compare *Kinematics*, Arts. 57, 68, 291 sq.).

It follows that concurrent forces, for instance, can be com-

pounded by geometrical addition, as will be explained more fully in Chapter IV.

On the other hand, kinetic energy and work are not vector-quantities.

74. The ideas of momentum, force, energy, work, with the fundamental equations connecting them, as given in the preceding articles, form the groundwork of the whole science of theoretical dynamics. The application of this science to the interpretation of natural phenomena gives results in exact agreement with observation and experiment. It is therefore important to inquire what are the physical assumptions and experimental data on which this application of dynamics is based.

These assumptions were formulated with remarkable clearness by Sir Isaac Newton in his *Philosophiæ naturalis principia mathematica*, first published in 1687, and have since been known as **Newton's laws of motion**. As these three *axiomata sive leges motus*, as Newton terms them, are very often referred to and, at least by English writers on dynamics, are usually laid down as the foundation of the science,* they are given here in a literal translation :

I. Every body persists in its state of rest or of uniform motion along a straight line, except in so far as it is compelled by impressed (*i.e.* external) forces to change that state.

II. Change of motion is proportional to the impressed moving force and takes place along the straight line in which that force acts.

III. To every action there is an equal and contrary reaction ; or, the mutual actions of two bodies on one another are always equal and directed in contrary senses.

75. Some explanation is necessary to correctly understand the meaning of these laws ; indeed, Newton's laws should not be studied by themselves. They become intelligible only if taken in connection with the definitions preceding them in the *Principia*, and with the explanations and corollaries that Newton himself has appended to them.

The word "body" must be taken to mean particle ; the word "motion" in the second law means what is now called momentum.

All three laws imply the idea of *force as the cause of any change of momentum* in a particle.

* See the *Syllabus of elementary dynamics*, Part I., London, Macmillan, 1890, p. 13 *sq.*, prepared by the Association for the Improvement of Geometrical Teaching.

76. With this definition of force the first law, at least in the ordinary form of statement, for a single particle, merely states that where there is no cause there is no effect. While this law may appear superfluous to us, it was not so in the time of Newton. Kepler and Galilei, less than a century before Newton, were the first to insist more or less clearly on this so-called **law of inertia**, viz. that there is no intrinsic power or tendency in moving matter to come to rest or to change its motion in any way.

77. The second law gives as the measure of a constant force the amount of momentum generated in a given time (see Art. 60); it can be called the **law of force**. If force be defined as the cause of any change of momentum, the second law follows naturally by assuming, as is always done, that the effect is proportional to the cause.

The first two laws may thus be regarded from the mathematical point of view as nothing but a definition of force; but they are certainly meant to emphasize the physical fact that the assumed definition of force is not arbitrary, but based on the characteristics of motion as observed in nature.

In the corollaries to his laws Newton shows how the composition and resolution of forces by the parallelogram rule follows from his definition. In deriving this result he tacitly assumes that the action of any force on a particle takes place independently of the action of any other forces that may be acting on the particle at the same time, a principle that would seem to deserve explicit statement. Some writers on mechanics, in particular French authors, prefer to replace Newton's second law by this *principle of the independence of the action of forces*.

78. The third law expresses the physical fact that in nature all forces occur in pairs of equal and opposite forces. In modern phraseology, two such equal and opposite forces in the same line are said to constitute a *stress*. Newton's third law is therefore called the **law of stress**.

This law, which was first clearly conceived in Newton's time, involves what may be regarded as the second fundamental property of matter or mass (the first being its indestructibility); viz. that *any two particles of matter determine in each other oppositely directed accelerations along the line joining them*.

79. For a more complete discussion of the physical laws underlying the applications of theoretical mechanics, the student is referred to THOMSON and TAIT, *Natural philosophy*, London, Macmillan, 1879,

Part I., Chapter II., p. 219 *sq.*; E. MACH, *Die Mechanik in ihrer Entwicklung*, Leipzig, Brockhaus, 1889, p. 203; K. PEARSON, *The grammar of science*, London, Scott, 1892, p. 357 *sq.*; J. D. EVERETT, *C.G.S. system of units*, London, Macmillan, 1891, p. 73; P. G. TAIT, article, "Mechanics," in the *Encyclopædia Britannica*, 9th ed.; J. CLERK MAXWELL, *Matter and Motion*, New York, Van Nostrand, 1878; P. G. TAIT, *Properties of matter*, Edinburgh, Black, 1885.

CHAPTER IV.

STATICS.

I. *Introduction.*

80. When a particle has two equal and opposite accelerations j , $-j$, its motion will not be changed. The same result must follow when a particle is acted on by two equal and opposite forces $F = mj$, $F' = -mj$. Their combined effect on the particle is *nil*, so that the particle, if originally at rest, will remain at rest; if originally moving with constant velocity in a straight line, it will continue to do so; and if originally moving under the action of any other forces in any way whatever, the introduction of the two equal and opposite forces will have no effect on its motion.

We say that two equal and opposite forces acting on a particle *balance*, or *are equivalent to 0*, or *are in equilibrium*. If no other forces act on the particle, the particle itself is said to be in equilibrium. It must be kept in mind that equilibrium is not synonymous with rest.

81. Let us next consider *any* two forces F_1 , F_2 acting simultaneously on the same particle P of mass m , and let j_1 , j_2 be the accelerations produced by these forces so that

$$F_1 = mj_1, \quad F_2 = mj_2.$$

The resultant acceleration of the particle is found by geometrically adding the vectors j_1 , j_2 ; let j be their geometric sum. Then the force

$$F = mj$$

producing the resultant acceleration is called the **resultant** of the forces F_1, F_2 ; these, or any other two or more forces having the same resultant F , are called the **components** of F .

82. In many investigations we are not so much concerned with the actual accelerations produced as with the effects that *might be* produced by any particular force or system of forces if the particle or body were perfectly free to move, *i.e.* not subject to other forces or restraints.

We proceed to study the composition and resolution of forces from this point of view, *i.e.* without reference to the accelerations produced, but with particular attention to the conditions under which the given system of forces is in equilibrium. This study forms the subject of **Statics**.

83. The geometrical characteristics of a force are (a) its *line* of action, (b) its *magnitude* or intensity, (c) its *sense*. Properly speaking, two forces should be called equal only when they agree in these three characteristics. But it is customary to call two forces *equal* even when they have only equal magnitude; we shall call them *geometrically equal*, when they agree in all three characteristics.

84. A force acting on a particle P is said to have its *point of application* at P , and the line representing it is usually drawn from P as origin. But the point of application is not an

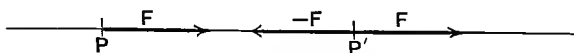


Fig. 14.

essential characteristic of the force; it may be taken at any point of its line if this line be regarded as rigid. Thus the force F acting on the particle P (Fig. 14) can be transferred, without changing its effect, to any point P' of its line; and two equal and opposite forces in the same line, such as F at P and

$-F$ at P' , are in equilibrium; provided always that P and P' may be regarded as belonging to the same rigid body.

85. It follows from Arts. 81 and 84 that any two forces F_1 F_2 whose lines intersect, say at O' (Fig. 15), are equivalent to,

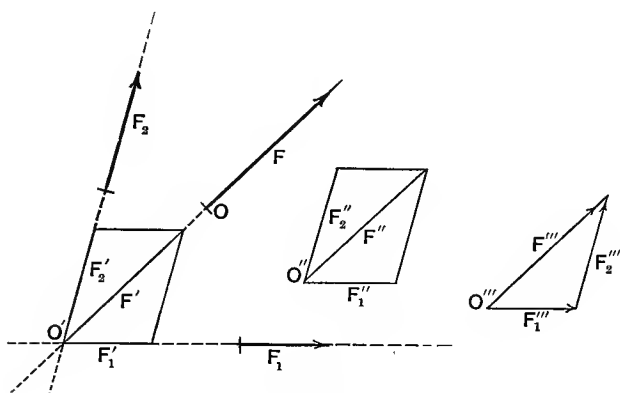


Fig. 15.

i.e. can be replaced by, a single force F called their *resultant*. This resultant can be found by replacing the forces F_1 , F_2 by the equal forces F_1' , F_2' at O' , and forming the parallelogram having F_1' , F_2' as adjacent sides. The diagonal F' through O' is the required resultant; it can be replaced by any force F of equal length and sense in the same line with this diagonal.

The parallelogram construction need not be made at O' ; we may select any origin O'' (Fig. 15), draw through it two vectors F_1'' , F_2'' equal (in direction, length, and sense) to F_1 , F_2 , find the diagonal F'' through O'' , and transfer it to a parallel line drawn through O' .

Finally, it is not necessary to draw the whole parallelogram; we have only to add the vectors F_1 , F_2 geometrically from any origin O''' (Fig. 15) and transfer their sum F''' to the parallel through O' .

86. Conversely, any force may be resolved into two components along any two lines intersecting the line of the force

at the same point and lying in the same plane with it. These components are together equivalent to the force, *i.e.* they may be substituted for the force.

87. It follows from Art. 85 that the resultant R of two intersecting forces P and Q , including the angle θ , is

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \theta}.$$

For two parallel forces or two forces acting in the same line, $\theta = 0$ or 180° , according as they are of equal or opposite sense; hence $R = P + Q$ in the former case, and $R = P - Q$ in the latter. It is also apparent that the resultant of any number of parallel forces or of forces acting in the same line is found as the algebraic sum of these forces. How the *position* of the resultant is found in the case of parallel forces will be shown later (Arts. 104, 106).

88. By Art. 86, to resolve a force R (Fig. 16) into two components P , Q along two lines making the angles α , β with the line of R , we have only to draw through the ends of a vector

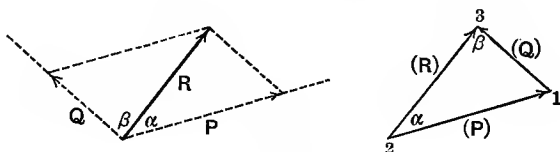


Fig. 16.

$23 = R$ lines 21 , 31 making angles α , β with 23 ; then $21 = P$, $13 = Q$. The triangle 123 gives the relations

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \alpha} = \frac{R}{\sin (\alpha + \beta)}.$$

When the components are at right angles, we have $P = R \cos \alpha$, $Q = R \sin \alpha$.

89. The projection of a closed polygon on any line being evidently zero, and the resultant being by definition the geo-

metric sum of its components, it follows that *the projection of the resultant on any line equals the algebraic sum of the projections of its components*. This proposition is sometimes expressed in the following form: the resolved part of the resultant in any direction is equal to the algebraic sum of the resolved parts of the components.

Let l be the line on which we project (Fig. 17), and let (l, R) (l, P) , (l, Q) denote the angles it makes with the resultant R and the components P , Q , respectively; then

$$R \cos (l, R) = P \cos (l, P) + Q \cos (l, Q).$$

90. Varignon's Theorem. Multiplying the last equation by any length $OS = s$ taken through the initial point O of R and at right angles to l , we obtain

$$R \cdot s \cos (l, R) = P \cdot s \cos (l, P) + Q \cdot s \cos (l, Q),$$

or since $s \cos (l, R) = r$, $s \cos (l, P) = p$, $s \cos (l, Q) = q$, where r , p , q are the perpendiculars let fall from S , on R , P , Q , respectively,

$$Rr = Pp + Qq.$$

In this form the proposition is independent of the direction of the line l and holds for any point S in the plane of the parallelogram.

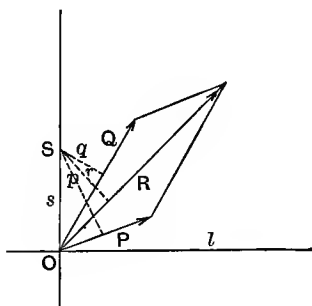


Fig. 17.

91. Moment of a Force. The product of a force into its perpendicular distance from a point is called the **moment** of the force about the point. It is taken with the positive or negative sign according as the force as seen from the point is directed counter-clockwise or clockwise.

The proposition of Art. 90, $Pp + Qq = Rr$, can now be stated in the following form: *the algebraic sum of the moments of any two intersecting forces about any point in their plane is equal to the moment of their resultant about the same point.*

92. The product Rr represents twice the area of the triangle having R for its base and S for its vertex; Pp , Qq can be interpreted similarly. This remark leads to another simple proof of Varignon's theorem, which may serve to make its meaning better understood. With the notation of Fig. 18 we have

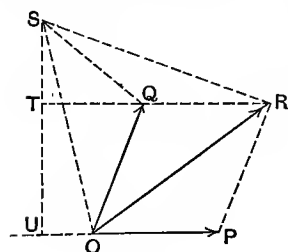


Fig. 18.

$$SOR = SOQ + SQR + QOR,$$

$$\text{or} \quad R \cdot r = Q \cdot q + P \cdot ST + P \cdot TU;$$

$$\text{or since } ST + TU = SU = p,$$

$$Rr = Qq + Pp.$$

93. If the point S be taken on the resultant R , we have $r=0$, hence $Pp = -Qq$; i.e. *the sum of the moments of two forces about any point on their resultant is zero.*

94. The forces of nature receive various special names according to the circumstances under which they occur. Thus the **weight** of a mass has already been defined (Art. 67) as the force with which the mass is attracted by the mass of the earth.

When a string carrying a mass at one end is suspended with its other end from a fixed point, it will be *stretched*, i.e. subjected to a certain tension. This means that if the string were cut it would require the application of a force along the line of the string to keep the weight in equilibrium. This force, which may thus serve to replace the action of the string, is called its **tension**.

When the surfaces of two physical bodies A , B are in contact, a **pressure** may exist between them; that is, if one of the bodies, say B , be removed, it may require the introduction of a force to keep A in the same state of rest or motion that it had before the removal of B . This force, which will obviously

act along the common normal of the surfaces at the point of contact, is called the **resistance** of B , and a force equal and opposite to it is called the **pressure** exerted by A on B .

95. Exercises.

(1) Find the resultant of two equal forces acting at right angles to each other.

(2) Show that the resultant R of two equal forces P including an angle θ is $R = 2P \cos(\theta/2)$.

(3) If the resultant of two equal forces be equal to P , what is the angle between the components?

(4) Find the magnitude and direction of the resultant of two forces of 100 and 200 lbs., including an angle of 60° .

(5) Let R be the effective piston pressure of a steam engine and ϕ the angle between the direction of motion of the piston and the connecting rod at any moment; show that the thrust in the connecting rod is $R \sec \phi$ and the pressure on the guide-bars $R \tan \phi$. For what position of the crank is the pressure on the guides greatest?

(6) A weight W is suspended from two fixed points A, B by means of a string ACB , C being the point of the string where the weight W is attached. If AC, BC be inclined to the vertical at angles α, β , find the tensions in AC, BC : (a) analytically; (b) graphically.

(7) Resolve a force of 20 lbs. into two components making angles of 45° and 30° with the given force: (a) analytically; (b) graphically.

(8) Find the rectangular components of a force P if one of the components is to make an angle of 30° with P .

(9) The resultant R , one of the components P , and the angle between the two components, $\theta = 60^\circ$, being given, find the other component Q .

(10) A particle is acted on by two forces P, Q lying in the same vertical plane and inclined to the horizon at angles p, q . Find their resultant in magnitude and direction, if $P = 527$ lbs., $Q = 272$ lbs., $p = 127^\circ 52'$, $q = 32^\circ 13'$.

(11) Prove that the moments of the two components of a force about any point on the line of the force are equal and opposite.

(12) Two forces acting on a point are represented in magnitude and direction by the tangent and normal of a parabola passing through the point. Find their resultant, and show that it passes through the focus of the parabola.

(13) The magnitudes of two forces acting on a point are as 2 to 3. If their resultant be equal to their arithmetic mean, what is the angle between the forces?

(14) What is the angle between a force of 1 ton and a force of $\sqrt{3}$ tons if their resultant is 2 tons?

(15) A string with equal weights W attached to its ends is hung over two smooth pegs A, B fixed in a vertical wall. Find the pressure on the pegs: (a) when the line AB is horizontal; (b) when it is inclined to the horizon at an angle θ . The weight of the string, its extensibility and stiffness, and the friction on the pegs are neglected in this problem as well as in those immediately following.

(16) The string being hung over three pegs A, B, C , determine graphically the pressures on the pegs. Let the vertical line through B lie between the vertical lines drawn through A and C ; there will be a pressure on B only if B lies above the line AC . If B lies below AC , the pressure may be distributed over the three pegs by passing the string around the peg B from below.

(17) In Ex. (15), for what position of the line AB are the pressures equal?

(18) In Ex. (16), let AC be horizontal, and let α, β, γ denote the angles of the triangle ABC . What are the pressures on the pegs?

(19) In Ex. (18), what must be the position of B to make the pressures on the three pegs equal: (a) when B lies above AC ; (b) when B lies below AC ?

(20) If the string with the equal weights W attached to its ends be strung over any number of pegs, the pressures on the pegs are readily determined, either graphically or analytically, in magnitude and direction; these pressures depend only on the value of W and on the angles between the successive sides of the polygon formed by the string, but not on the distances between the pegs.

(21) Suppose the string be closed, its ends being fastened together. Let this string be hung over three pegs A, B, C forming an isosceles triangle in a vertical plane with its base AC horizontal, and let a weight

W be suspended from the lowest point D of the string. If $AC=4$ ft., $AB=BC=2.5$ ft., and the length of the string $2l=14$ ft., find the tension of the string and the pressures on the pegs.

(22) If, in Ex. (21), the triangles ABC and ADC be equilateral, what would be the tension and the pressures on the pegs?

(23) In Ex. (21), the triangles ABC and ADC being isosceles and their common base AC horizontal, what must be the relation between the angles 2β at B and 2δ at D to make the pressures on the three pegs A, B, C equal? The pressures being made equal, what angle gives the least pressure?

(24) Show, both analytically and geometrically, that a force whose components P_1, P_2 make an angle θ can be resolved into two rectangular components $(P_1 + P_2) \cos(\theta/2), (P_1 - P_2) \sin(\theta/2)$.

(25) In the toggle-joint press two equal rods CA, CB are hinged at C ; a force F bisecting the angle 2α between the rods forces the ends A, B apart. If A be fixed, find the pressure exerted at B at right angles to F if $F=100$ lbs. and $\alpha=15^\circ, 35^\circ, 65^\circ, 85^\circ, 90^\circ$.

II. Concurrent Forces.

96. Let there be given any number n of forces $F_1, F_2, F_3, \dots, F_n$, whose directions all pass through the same point. By Art. 85, we can find the resultant R_1 of F_1 and F_2 , next the resultant R_2 of R_1 and F_3 , then the resultant R_3 of R_2 and F_4 , and so on. The resultant R of R_{n-2} and F_n is evidently equivalent to the whole system $F_1, F_2, F_3, \dots, F_n$ and is called its **resultant**. We thus have the proposition that *a system consisting of any number of concurrent forces is equivalent to a single resultant*.

97. It may of course happen that this resultant is zero. In this case, the system is said to be *in equilibrium*. The condition of equilibrium of a system of concurrent forces is therefore $R=0$.

98. In practice, the process of finding the resultant indicated in Art. 96 is inconvenient when the number of forces is large.

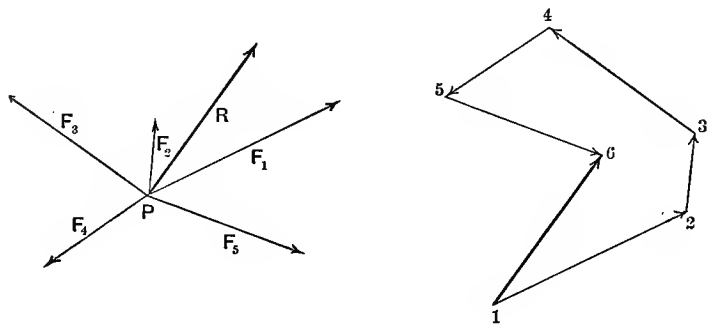


Fig. 19.

If the forces are given graphically, by their vectors, we have only to add these vectors geometrically (see *Kinematics*, Art. 46), and this can best be done in a separate diagram, called the **force polygon**, or **stress diagram**. Thus, in Fig. 19, 1 2 is drawn equal and parallel to F_1 , 2 3 equal and parallel to F_2 , 3 4 to F_3 , 4 5 to F_4 , 5 6 to F_5 . The closing line of the force polygon, viz. 1 6 in

the figure, is equal and parallel to the resultant R , which is therefore obtained by drawing through the point of intersection of the forces a line equal and parallel to 16.

The graphical condition of equilibrium consists in the closing of the force polygon, that is, in the coincidence of its terminal point (6) with its initial point (1).

99. Analytically, a systematic solution is obtained by resolving each force F into three components X, Y, Z , along three rectangular axes passing through the point of intersection of the given forces. All components lying in the direction of the same axis can then be added algebraically, and the whole system of forces is found to be equivalent to three rectangular forces $\Sigma X, \Sigma Y, \Sigma Z$, which, by the parallelogram law, can be combined into a single resultant

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.$$

The angles α, β, γ made by this resultant with the axes are given by the relations

$$\frac{\cos \alpha}{\Sigma X} = \frac{\cos \beta}{\Sigma Y} = \frac{\cos \gamma}{\Sigma Z} = \frac{1}{R}.$$

100. If the forces all lie in the same plane, only two axes are required, and we have

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}, \quad \tan \theta = \frac{\Sigma Y}{\Sigma X},$$

where θ is the angle between the axis of X and R .

101. The condition of equilibrium (Art. 97) $R=0$ becomes, by Art. 99, $(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 = 0$. As all terms in the left-hand member are positive, their sum can vanish only when each term is $=0$. *The analytical conditions of the equilibrium of any number of concurrent forces* are therefore :

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0.$$

102. As the projection on any line of any closed polygon, even when its sides do not all lie in the same plane, is equal to 0, it follows that the proposition of Art. 89 holds for any number of concurrent forces.

103. Exercises.

(1) Show that three forces that are in equilibrium must lie in the same plane and pass through the same point.

(2) Six forces of 1, 2, 3, 4, 5, 6 lbs., respectively, act in the same plane on the same point, making angles of 60° with each other. Find their resultant in magnitude and direction : (a) graphically ; (b) analytically.

(3) Let $AB = c$ (Fig. 20) be the vertical post, $AC = b$ the jib, of a crane, the ends BC being connected by a chain of length a . If a weight W be suspended from C , find the tension T produced by it in the chain and the thrust P in AC .

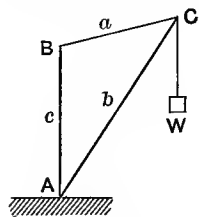


Fig. 20.

(4) Let AC be hinged at A (Fig. 20) so as to turn freely in a vertical plane, and let the chain pass over a pulley at C and carry the weight W . In what position of AC will there be equilibrium?

(5) Find the resultant R of three concurrent forces A, B, C lying in the same plane and making angles α, β, γ with each other.

(6) Prove that the moment of the resultant of any number of concurrent forces lying in the same plane about any point in this plane is equal to the sum of the moments of the forces about the same point.

(7) By means of Ex. (6), express the conditions of equilibrium of any number of concurrent forces in the same plane.

(8) When three forces are in equilibrium, show that they are proportional and parallel to the sides of a triangle.

(9) When any number of concurrent forces are in equilibrium, show that any one of them reversed is the resultant of all the others.

(10) A weightless rod AC (Fig. 21), hinged at one end A so as to be free to turn in a vertical plane, is held in a horizontal position by means of the chain BC . If a weight W be suspended at C , find the thrust P in AC and the tension T of the chain. Assume $AC = 8$ ft., $AB = 6$ ft.

(11) In Ex. (10), suppose the rod AC , instead of being hinged at A , to be set firmly into the wall in a horizontal position; and let the chain fastened at B run at C over a smooth pulley and carry the weight W . Find the tension of the chain and the magnitude and direction of the pressure on the pulley at C .

(12) In "tacking against the wind," let W be the force of the wind; α, β the angles made by the axis of the boat with the direction in which the wind blows, and with the sail, respectively. Determine the force that drives the boat forward and find for what position of the sail it is greatest.

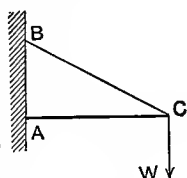


Fig. 21.

(13) A cylinder of weight W rests on two inclined planes whose intersection is horizontal and parallel to the axis of the cylinder. Find the pressures on these planes.

(14) Find the tensions in the string $ABCD$, fixed at A and D , and carrying equal weights W at B and C , if $AD = c$ is horizontal, $AB = BC = CD$, and the length of the string is 3ℓ .

(15) One of the vertices A of a regular hexagon is acted upon by 5 forces represented in magnitude and direction by the lines drawn from A to the other vertices of the hexagon. Find their resultant.

(16) Find the resultant of three equal forces P acting on a point, the angle between the first and second as well as that between the second and third being 45° .

(17) A mass m rests on a plane inclined to the horizon at an angle θ ; it is kept in equilibrium (a) by a force P_1 parallel to the plane; (b) by a horizontal force P_2 ; (c) by a force P_3 inclined to the horizon at an angle $\theta + \alpha$. Determine in each case the force P and the pressure R on the plane.

(18) Show that the three forces represented by the vectors OA, OB, OC are in equilibrium if O is the centroid of the triangular area ABC .

(19) Show that the three vectors OA , OB , OC have the same resultant as the three vectors OA' , OB' , OC' , if A' , B' , C' are the middle points of the sides of the triangle ABC .

(20) Show that the resultant of the vectors OA , OB , OC is OO' , if O is the centre of the circle circumscribed to the triangle ABC and O' the intersection of the altitudes of the same triangle.

III. *Parallel Forces.*

104. Resultant of Two Parallel Forces. The graphical construction of the resultant (Art. 85) fails in the case of parallel forces.

As an expedient, we may resolve one of the two given forces into two components and then combine these successively with the other force. Thus, resolving P (Fig. 22) into P' and P'' along the lines I and II respectively, we may compound P'' with Q , and their resultant (acting along III) with P' . The resolution of P into two arbitrary components P' , P'' is best done in a separate diagram, the **force polygon**, by taking 1 2 equal and parallel to P , and drawing from any arbitrary point O ,

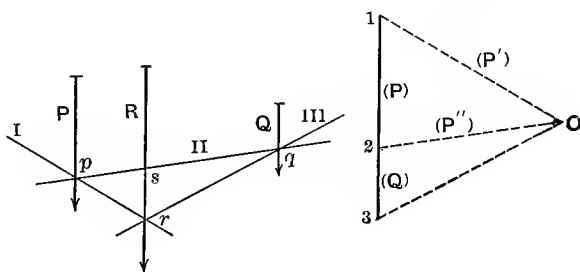


Fig. 22.

called the **pole**, O_1 , O_2 , which will represent the components P' , P'' in magnitude and direction. Then drawing 2 3 equal and parallel to Q , we find O_3 as the resultant of P'' and Q .

The whole operation of finding the resultant R of two parallel forces P , Q is therefore as follows. First construct the *force polygon* by making 1 2 equal and parallel to P , 2 3 equal and parallel to Q ; 1 3 gives the magnitude and direction of the resultant R . Then assume a pole O and draw O_1 , O_2 , O_3 . Now construct the so-called **funicular polygon** (or **equilibrium polygon**) by drawing in the original figure a line I parallel to O_1 intersecting P say in p ; through p a line II parallel to O_2 in-

intersecting Q say in q ; through q a line III parallel to $O3$. The intersection r of I and III is a point of the resultant R which is therefore obtained in position by drawing through r a line equal and parallel to 13.

105. In Fig. 22 the two given parallel forces P , Q were assumed of the same sense. The construction applies, however, equally well to the case when they are of opposite sense. The resultant R will then be found to lie not between P and Q , but outside, on the side of the larger force. The construction fails only when the two given forces are equal and of opposite sense, a case that will be considered later (see Art. 112 and Arts. 128–138).

106. To determine the position of R analytically, we may find the ratio in which it divides the distance (perpendicular or oblique) between P and Q . Let s (Fig. 22) be the point where R meets pq . Then, since the triangles prs and $O12$, as well as the triangles qsr and $O23$, are similar, we have

$$\frac{ps}{sr} = \frac{O2}{P}, \quad \frac{sq}{sr} = \frac{O2}{Q};$$

hence, dividing,
$$\frac{ps}{sq} = \frac{Q}{P}.$$

This means that *the resultant of two parallel forces divides their distance in the inverse ratio of the forces*. As this proposition finds application in the theory of the lever, it is commonly referred to as the **principle of the lever**.

Dropping perpendiculars p , q from any point of the resultant R on the components P , Q , the relation can be expressed in the form

$$Pp = -Qq,$$

which shows that Varignon's proposition of moments (Arts. 89–93) applies to parallel forces.

107. The resultant of two parallel forces can also be found by the following simple process. Intersect the two parallel forces P , Q by any transversal in p and q (Fig. 23) and apply at these points along pq two equal and opposite forces F , $-F$; find the resultant P' of F and P and the resultant P'' of $-F$ and Q ; these resultants P' and P'' will intersect (unless P and Q be equal and opposite) and their resultant R can be found.

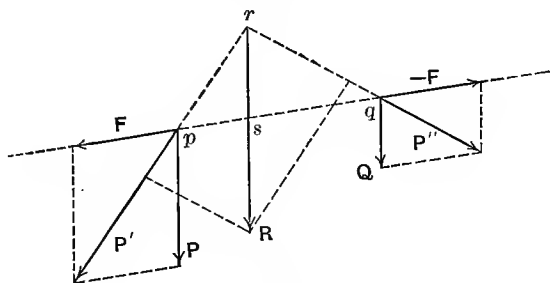


Fig. 23.

It will be noticed that this construction reduces to that given in Art. 104 if for F we select the force $2O$ in the force polygon, Fig. 22, p. 59.

108. Resultant of Any Number of Parallel Forces. The graphical method of Art. 104 is readily extended to the general case of any number of parallel forces lying in the same plane. Whatever the number of the forces, the force polygon gives magnitude, direction, and sense of the resultant, which is simply the algebraic sum of the given forces; while the funicular polygon (formed by the lines I, II, III, etc.) gives the position of the resultant by furnishing one of its points, viz. the intersection of the first and last sides of the funicular polygon.

The process will best be understood from the following example.

The horizontal beam AB (Fig. 24) resting freely on the fixed supports A , B carries four weights W_1 , W_2 , W_3 , W_4 .

To determine the position of the resultant and the reactions A , B of

$A'I$ and $IV B'$. The intersection of these lines, *i.e.* of the first and last sides of the funicular polygon, gives a point, R , of the resultant of W_1, W_2, W_3, W_4 .

Moreover, if the component in $A'I$ be resolved along $A'B'$ and the vertical through A' , and similarly the component in $B'IV$ along $B'A'$ and the vertical through B' , the two components along $A'B'$ will be equal and opposite, each being equal to the parallel Oo drawn to $A'B'$ in the force polygon. This parallel furnishes, therefore, the magnitudes of the reactions $A = 0.1, B = 5.0$.

109. Analytically, the resultant of n parallel forces $F_1, F_2, \dots F_n$, whether in the same plane or not, can be found as follows:

The resultant of F_1 and F_2 is a force $F_1 + F_2$ situated in the plane (F_1, F_2), so that $F_1 p_1 = F_2 p_2$ (Art. 106), where p_1, p_2 are the (perpendicular or oblique) distances of the resultant from F_1 and F_2 , respectively. This resultant $F_1 + F_2$ can now be combined with F_3 to form a resultant $F_1 + F_2 + F_3$, whose distances from $F_1 + F_2$ and F_3 in the plane determined by these two forces are as F_3 is to $F_1 + F_2$. This process can be continued until all forces have been combined; the final resultant is

$$F_1 + F_2 + \dots + F_n.$$

Any number of parallel forces are, therefore, in general equivalent to a single resultant equal to their algebraic sum.

110. To find the *position* of this resultant analytically, let the points of application of the forces $F_1, F_2, \dots F_n$ be $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots (x_n, y_n, z_n)$. The point of application of the resultant $F_1 + F_2$ of F_1 and F_2 may be taken so as to divide the distance of the points of application of F_1 and F_2 in the ratio $F_2 : F_1$; hence, denoting its co-ordinates by x', y', z' , we have $F_1 (x' - x_1) = F_2 (x_2 - x')$, or

$$(F_1 + F_2) x' = F_1 x_1 + F_2 x_2,$$

and similarly for y' and z' .

The force $F_1 + F_2$ combines with F_3 to form a resultant $F_1 + F_2 + F_3$, whose point of application x'', y'', z'' is given by

$$(F_1 + F_2 + F_3) x'' = F_1 x_1 + F_2 x_2 + F_3 x_3,$$

and similar expressions for y'', z'' .

Proceeding in this way, we find for the point of application $(\bar{x}, \bar{y}, \bar{z})$ of the resultant of all the given forces

$$(F_1 + F_2 + \dots + F_n) \bar{x} = F_1 x_1 + F_2 x_2 + \dots + F_n x_n,$$

with corresponding equations for $\bar{y}$ and $\bar{z}$. We may write these equations in the form :

$$\bar{x} = \frac{\sum Fx}{\sum F}, \quad \bar{y} = \frac{\sum Fy}{\sum F}, \quad \bar{z} = \frac{\sum Fz}{\sum F}. \quad (1)$$

As these expressions for $\bar{x}, \bar{y}, \bar{z}$ are independent of the direction of the parallel forces, it follows that the same point $(\bar{x}, \bar{y}, \bar{z})$ would be found if the forces were all turned in any way about their points of application, provided they remain parallel. The point $(\bar{x}, \bar{y}, \bar{z})$ is for this reason called the **centre** of the system of parallel forces. It is nothing but what in geometry is called the *mean point*, or *mean centre*, of the points of application if the forces are regarded as coefficients or "weights" (in the meaning of the theory of least squares) of these points.

111. As the origin of co-ordinates in the last article is arbitrary, the equations (1) evidently express the proposition that *in any system of parallel forces the sum of their moments about any point is equal to the moment of their resultant about the same point*. In particular, *the sum of the moments about any point on the resultant is zero*.

This proposition may be regarded as a generalisation of the *principle of the lever* referred to in Art. 106. It furnishes the convenient method of "taking moments" for the purpose of determining the position of the resultant.

112. Couple of Forces. The construction given in Art. 104 for the resultant of two parallel forces fails only when the two

given forces are equal and of opposite sense. In this case, the lines I and III of the funicular polygon become parallel, so that their intersection r lies at infinity. The magnitude of the resultant is of course $=0$.

The combination of two equal and opposite parallel forces $(F, -F)$ is called a **couple**. A couple is, therefore, properly speaking, *not equivalent to a single force*, although it may be said to be equivalent to a force of magnitude 0 at an infinite distance. The theory of couples will be considered in detail in Arts. 128-138.

113. Conditions of Equilibrium. We have seen (Art. 109) that a system of n parallel forces is, in general, equivalent to a single force; but, as appears from the preceding article, it may happen to reduce to a couple. It follows that *for the equilibrium of a system of parallel forces the condition $R=0$, though always necessary, is not sufficient*.

Now, if the resultant R of the n parallel forces $F_1, F_2, \dots F_n$ be $=0$, the resultant R' of the $n-1$ forces $F_1, F_2, \dots F_{n-1}$ cannot be 0, and its point of application is found (by Art. 110) from $\bar{x} = (F_1x_1 + F_2x_2 + \dots + F_{n-1}x_{n-1}) / (F_1 + F_2 + \dots + F_{n-1})$ and similar expressions for $\bar{y}$ and $\bar{z}$. The whole system of parallel forces is therefore equivalent to the two parallel forces R' and F_n . Two such forces can be in equilibrium only when they lie in the same straight line; *i.e.* F_n must coincide with R' and must therefore pass through the point $(\bar{x}, \bar{y}, \bar{z})$, which is a point of R' .

The additional condition of equilibrium is, therefore,

$$\frac{\bar{x} - x_n}{\cos \alpha} = \frac{\bar{y} - y_n}{\cos \beta} = \frac{\bar{z} - z_n}{\cos \gamma},$$

where α, β, γ are the angles made by the direction of the forces with the axes.

114. For practical application it is usually best to replace the last condition by taking moments about a convenient point.

Thus, the analytical condition of equilibrium can be written in the form

$$\Sigma F=0, \quad \Sigma Fp=0.$$

Graphically, to the former corresponds the closing of the force-polygon, to the latter the closing of the funicular polygon.

115. Weight; Centre of Gravity. The most important special case of parallel forces is that of the force of gravity which acts at any given place near the earth's surface in approximately parallel lines on every particle of matter.

If g be the acceleration of gravity, the force of gravity on a particle of mass m is

$$w=mg,$$

and is called the **weight** of the particle or of the mass m .

For a system of particles of masses $m_1, m_2, \dots m_n$ we have

$$w_1=m_1g, \quad w_2=m_2g, \quad \dots \quad w_n=m_ng.$$

The resultant W of these parallel forces,

$$W=w_1+w_2+\dots+w_n=(m_1+m_2+\dots+m_n)g=Mg,$$

where M is the mass of the system, is called the weight of the system.

The centre of the parallel forces of gravity of a system of particles has, by Art. 110, the co-ordinates

$$\bar{x}=\frac{\Sigma mgx}{\Sigma mg}, \quad \bar{y}=\frac{\Sigma mgy}{\Sigma mg}, \quad \bar{z}=\frac{\Sigma mgz}{\Sigma mg},$$

or since the constant g cancels,

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}, \quad \bar{y}=\frac{\Sigma my}{\Sigma m}, \quad \bar{z}=\frac{\Sigma mz}{\Sigma m}.$$

This point is called the **centre of gravity** of the system, and is evidently identical with the *centre of mass*, or **centroid** (see Art. 13).

For continuous masses the same formulæ hold, except that the summations become integrations.

The *weight* W of a physical body of mass M is therefore a vertical force passing through the centroid of its mass.

116. Exercises.

(1) A straight rod (*lever*) of length $2l = 5$ ft. has suspended from its ends masses of 12 and 27 pounds, respectively. Find the point (*fulcrum*) on which it balances in a horizontal position: (a) if its own weight be neglected; (b) if it is homogeneous and weighs 2.2 pounds per running foot.

(2) A straight beam rests in a horizontal position on two supports A, B . The distance between the supports (the *span*) is $2l = 24$ ft. The beam carries a weight of 14 tons at a distance of 8 ft. from A , and a weight of 10 tons at 16 ft. from A . Find the pressures on the supports (or the *reactions* of the supports): (a) when the proper weight of the beam is neglected; (b) when the beam weighs $\frac{1}{4}$ ton per running foot; (c) when the first third of the beam (from A) weighs $\frac{1}{8}$ ton, the second 1 ton, the third $\frac{1}{2}$ ton per running foot.

(3) A homogeneous circular plate weighing W pounds rests in a horizontal position on three equidistant supports near its edge. (a) What is the least weight P that will upset it when placed on the plate? (b) If there be four equidistant supports near the edge, what is the least weight that will upset the plate?

(4) Construct the resultant of two parallel forces of opposite sense by the graphical method of Arts. 104, 105.

(5) Solve exercises (1) and (2) by the graphical method.

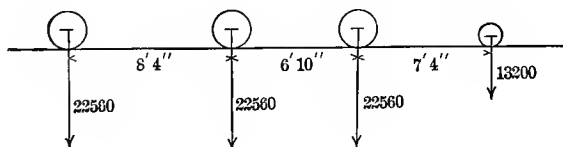


Fig. 25.

(6) Find the reactions of the supports of a bridge truss of 50 ft. span, produced by a freight locomotive whose weight is distributed over the three pairs of driving wheels and the front truck, as indicated in Fig. 25: (a) when it stands in the middle of the span; (b) when its front truck stands over one support.

(7) Explain how the centroid of a plane area can be found graphically by dividing the area into narrow parallel strips.

(8) A homogeneous rectangular plate is pivoted on a horizontal axis through its centre so as to turn freely in a vertical plane. If weights W_1, W_2, W_3, W_4 be suspended from its vertices, what is its position of equilibrium?

(9) The ends of a straight lever of length l are acted upon by two forces F_1, F_2 in the same plane with it, but inclined to the lever at angles α_1, α_2 . Determine the position of the fulcrum.

117. Funicular Polygons and Catenaries. The *funicular polygon* in its original meaning represents the form of equilibrium assumed by a string or cord suspended from two fixed points and acted upon by any forces in the same plane. The "cord" is supposed to be perfectly flexible, inextensible, inelastic, and without weight. When the number of forces is made infinite, the polygon becomes a continuous curve called a *catenary*.

The present discussion is confined to the case when the forces are all vertical so that they can be regarded as weights.

118. Let A, B (Fig. 26) be the fixed points, and let there be five weights, W_1, W_2, W_3, W_4, W_5 , suspended from the points I, II, III, IV, V, of the cord.

If the cord be cut on both sides of the point I and the corresponding tensions T_1, T_2 be introduced, the point I must be in equilibrium under the action of the three forces W_1, T_1, T_2 . Hence drawing a line 1 2 to represent the weight W_1 and drawing through its ends 1, 2 parallels to AI and $I II$, respectively, we have the force polygon of the point I. Its sides $O 1$ and $2 O$ represent in magnitude, direction, and sense the tensions T_1, T_2 ; in other words, the weight W_1 has thus been resolved into its components along the adjacent sides.

The same can be done at every vertex of the polygon I II III IV V, and all tensions can thus be found. But as the tension T_2 in I II occurs again (with sense reversed) in the force polygon for the point II, and so on, the successive force polygons can be fitted together, every triangle having one

side in common with the next one. Thus the complete force polygon of the whole cord is formed, as shown on the right in Fig. 26. Its vertical line represents the successive weights $W_1=1\ 2$, $W_2=2\ 3$, $W_3=3\ 4$, $W_4=4\ 5$, $W_5=5\ 6$, while the lines

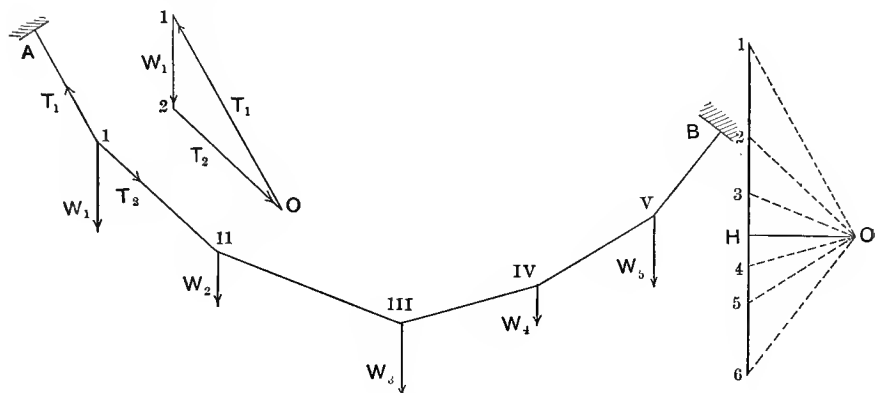


Fig. 26.

radiating from the point O , or pole, represent on the same scale the tensions in AI , $I\ II$, $II\ III$, $III\ IV$, $IV\ V$, $V\ B$.

119. The polygon $AI\ II\ III\ IV\ V\ B$ is called the *funicular polygon*. It will be noticed that if we have given the fixed points A , B , the magnitudes of the weights, their horizontal distances, say from A , and the directions of the first and last sides AI , VB (whatever may be the number of the forces), the remaining sides of the funicular polygon can be found by laying off on a vertical line the weights $W_1=1\ 2$, $W_2=2\ 3$, etc., in succession, drawing through 1 a parallel to the first side, through the end of the last weight (6 in Fig. 26) a parallel to the last side, and joining the intersection O of these parallels to the points 2, 3, etc. The sides of the funicular polygon must be parallel to the lines radiating from O ; at the same time these lines represent the tensions in these sides.

120. For the analytical investigation, let P_i be that vertex of a funicular polygon of any number of sides at which the i th and

$(i+1)$ th sides intersect; let α_i, α_{i+1} be the angles at which these sides are inclined to the horizon, and W_i the weight suspended from the vertex P_i (Fig. 27).

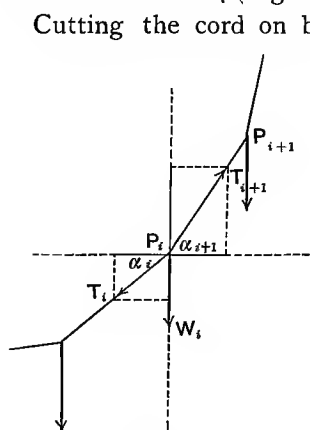


Fig. 27.

Cutting the cord on both sides of P_i , and introducing the tensions T_i and T_{i+1} , the conditions of equilibrium of the point P_i are found by resolving the three forces W_i, T_i, T_{i+1} horizontally and vertically (Art. 100):

$$T_{i+1} \cos \alpha_{i+1} = T_i \cos \alpha_i, \quad (1)$$

$$T_{i+1} \sin \alpha_{i+1} = T_i \sin \alpha_i + W_i. \quad (2)$$

The former of these equations shows that, whatever the weights W and the lengths and inclinations of the sides, *the horizontal components of the tensions T are all equal*. Denoting this constant value by H , we have

$$T_1 \cos \alpha_1 = T_2 \cos \alpha_2 = \dots = T_i \cos \alpha_i = \dots = H. \quad (3)$$

Substituting the values of T_i and T_{i+1} as obtained from these relations, into (2), this equation becomes

$$\tan \alpha_{i+1} = \tan \alpha_i + \frac{W_i}{H}, \quad (4)$$

which shows that as soon as all the weights and the inclination and tension of any one side are given, the inclinations and tensions of all the other sides can be found.

121. Let us now assume that the weights W are all equal. Then the values of $\tan \alpha_{i+1}$ given by (4) form an arithmetical progression. If, in addition, we assume that *the sides of the polygon are such as to have equal horizontal projections, i.e.* if we assume the weights to be equally spaced horizontally,

the vertices of the polygon will lie on a parabola whose axis is vertical.

To find its equation, let us suppose, for the sake of simplicity, that one side of the polygon, say the k th, is horizontal so that $\alpha_k = 0$. Taking this side of axis x , its middle point O as origin, the co-ordinates of the vertex P_k are $\frac{1}{2}a, 0$, if a be the length of the horizontal side and hence also that of the horizontal projection of every side.

Putting $W/H = \tau$, we have $\tan \alpha_k = 0$, $\tan \alpha_{k+1} = \tau$, $\tan \alpha_{k+2} = 2\tau$, ...; hence the co-ordinates of P_{k+1} are $x = \frac{3}{2}a$, $y = a\tau$; those of P_{k+2} are $x = \frac{5}{2}a$, $y = a\tau + 2a\tau = 3a\tau$; those of P_{k+3} are $x = \frac{7}{2}a$, $y = a\tau + 2a\tau + 3a\tau = 6a\tau$, etc.; those of the n th vertex after P_k are

$$x = \frac{2n+1}{2}a, \quad y = \frac{n(n+1)}{2}a\tau.$$

Eliminating n , we find the equation

$$x^2 = \frac{2a}{\tau} \left(y + \frac{a\tau}{8} \right),$$

which represents a parabola whose axis is the axis of y , and whose vertex lies at the distance $\frac{1}{8}a\tau = \frac{1}{8}aW/H$ below the origin O .

122. Let the number of sides be increased indefinitely, the length a and the weight W approaching the limit 0, but so that the quotient a/W remains finite, say $\lim (a/W) = 1/w$. Then $\lim (a/\tau) = H/w$, $\lim (a\tau) = 0$; so that the equation of the parabola becomes

$$x^2 = \frac{2H}{w}y,$$

where w is evidently the weight of the cord, or chain, per unit length.

The parabola is, therefore, the form of equilibrium of a cord suspended from two points when the weight of the cord is uniformly distributed over its horizontal projection. This is, for

instance, the case approximately in a suspension bridge with uniformly loaded roadbed, the proper weight of the chains being neglected.

123. This result can easily be derived independently of Art. 121, by considering the equilibrium of any portion OP of the chain beginning at the lowest point O (Fig. 28). The forces acting on this portion are the horizontal tension H at O , the tension T along the tangent at P , and the proper weight W of the chain. As this weight is assumed to be uniformly distributed over the horizontal projection $OP'=x$ of OP , the weight is $W=wx$, and bisects OP' .

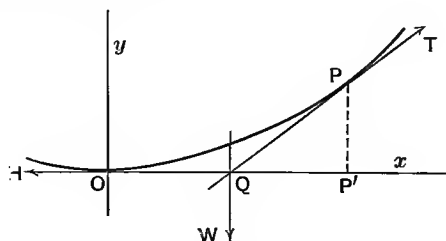


Fig. 28.

Resolving the forces in the horizontal and vertical directions, we find, as conditions of equilibrium,

$$-H + T \frac{dx}{ds} = 0, \quad -wx + T \frac{dy}{ds} = 0;$$

whence, eliminating ds ,

$$\frac{dy}{dx} = \frac{w}{H} x.$$

Integrating and considering that $x=0$ when $y=0$, we find the equation of the parabola as above,

$$y = \frac{w}{2H} x^2.$$

124. The three forces H , T , W are in equilibrium; they must intersect in a point Q which bisects OP' , and the force polygon must be similar to the triangle QPP' .

Hence, if the height of a suspension bridge be h , its span $2l$, its total weight $2W$, we have for the horizontal tension H , and the tension T at the point of support

$$\frac{H}{l/2} = \frac{T}{\sqrt{h^2 + l^2/4}} = \frac{W}{h}.$$

125. *The form of equilibrium assumed by a homogeneous cord is an ordinary catenary.*

To find its equation, we again consider the equilibrium of a

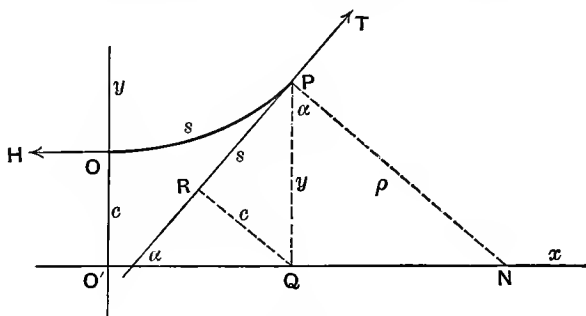


Fig. 29.

portion $OP=s$ (Fig. 29) of the cord, beginning at the lowest point O .

The weight of this portion is now $W=ws$, and if α be the angle made by the tangent at P with a horizontal line, we have the conditions of equilibrium

$$T \cos \alpha = T \frac{dx}{ds} = H, \quad T \sin \alpha = T \frac{dy}{ds} = ws.$$

Dividing and putting $H/w=c$, we have the differential equation of the curve in the form

$$\frac{dx}{dy} = \frac{c}{s}.$$

Substituting this value of dx/dy in the relation $ds^2 = dx^2 + dy^2$, we obtain

$$\left(\frac{ds}{dy}\right)^2 = 1 + \frac{c^2}{s^2}, \quad \text{or} \quad dy = \pm \frac{s ds}{\sqrt{s^2 + c^2}},$$

which gives by integration $y+c=\sqrt{s^2+c^2}$, the minus sign being rejected since y increases with s .

The constant C can be made to disappear by taking the origin O' on the vertical through O at the distance $O'O=c$ below the lowest point O . We have, therefore,

$$y^2=s^2+c^2.$$

By means of this relation, s can be eliminated from the original differential equation, and the result,

$$\frac{cdy}{\sqrt{y^2-c^2}}=dx,$$

can be integrated :

$$c \log (y + \sqrt{y^2 - c^2}) = x + C.$$

As $y=c$ when $x=0$, we find $C=c \log c$; hence

$$y + \sqrt{y^2 - c^2} = ce^{\frac{x}{c}}.$$

Taking reciprocals and rationalising the denominator, we find

$$y - \sqrt{y^2 - c^2} = ce^{-\frac{x}{c}};$$

hence, adding and subtracting,

$$y = \frac{c}{2} (e^{\frac{x}{c}} + e^{-\frac{x}{c}}), \quad s = \frac{c}{2} (e^{\frac{x}{c}} - e^{-\frac{x}{c}}).$$

126. The first equations of Art. 125, $T \cos \alpha = H = wc$, $T \sin \alpha = ws$, give for the total tension T at any point P

$$T^2 = w^2(c^2 + s^2) = (wy)^2$$

Thus, *while the horizontal component is constant, the vertical component at any point P is equal to the weight of the portion of the cord from the lowest point O to the point P, and the total tension is equal to the weight of a portion of the cord equal to the ordinate of the point P.*

Let Q be the foot of the ordinate of P (Fig. 29), N the intersection of the normal with the axis $O'x$, and draw QR perpendicular to the tangent. Then $PR = y \sin \alpha = s$, since $T \sin \alpha = ws$ and $T = wy$; also $QR = y \cos \alpha = c$. Dividing, we have $\tan \alpha = s/c$; hence, differentiating,

$$\frac{1}{\cos^2 \alpha} \frac{d\alpha}{ds} = \frac{1}{c}; \quad \therefore \rho = \frac{ds}{d\alpha} = \frac{c}{\cos^2 \alpha}.$$

The figure shows that the radius of curvature ρ is equal to the length of the normal PN .

The relation $\rho \cos^2 \alpha = c$ shows further that at the vertex ($\alpha = 0$) the radius of curvature is $\rho_0 = c$. It follows that for a cord or chain suspended from two points B, C in the same horizontal line, c (and consequently H) is large when ρ_0 is large, *i.e.* when the curve is flat at the vertex; in other words, when B and C are far apart.

127. Exercises.

(1) A weightless cord $ABCDEF$ is suspended from the fixed points A, F , and carries weights at the intermediate points B, C, D, E . Taking A as origin, the axis of x horizontal, the axis of y vertically upwards, the co-ordinates of the points B, C, D, E, F are $(2, -1), (4, -1.5), (7, -1.5), (8.5, -1), (10, 2)$. If the weight at B be one pound, what are the weights at C, D, E ? What are the tensions of the sections of the cord? What are the reactions of the fixed points A, F ?

(2) The total weight of a suspension bridge is $2W = 50$ tons; the span is $2l = 200$ ft.; the height is $h = 18$ ft. Find the tension of the chain at the ends and in the middle, both graphically and analytically.

(3) A uniform wire of length $2s$ is stretched between two points in the same horizontal line whose distance $2x$ is very nearly equal to $2s$. Find an approximate expression for the parameter c of the catenary and thence for the tension of the wire.

IV. *Theory of Couples.*

128. *The combination of two equal forces of opposite sense F , $-F$, acting along parallel lines, is called a couple of forces, or simply a **couple** (Art. 112).*

The perpendicular distance $AB=p$ (Fig. 30) of the forces of the couple is called the **arm**, and the product Fp of the force F into the arm p is called the **moment** of the couple.

If we imagine the couple (F, p) to act upon an invariable plane figure in its plane, and if the middle point of its arm be a fixed point of this figure, the couple will evidently tend to turn the figure about this middle point. (It is to be observed that it is *not* true, in general, that a couple acting on a rigid body produces rotation about an axis at right angles to its plane.) A couple of the type (F, p) or (F', p')

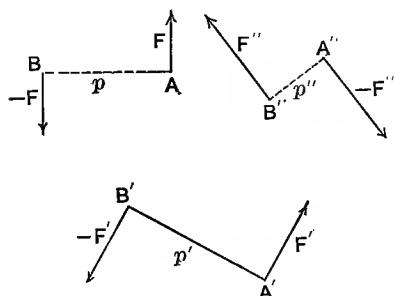


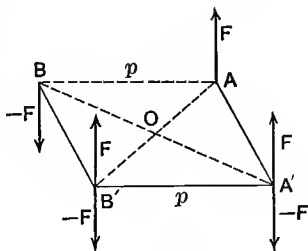
Fig. 30.

(see Fig. 30) will tend to rotate counter-clockwise, while a couple of the type (F'', p'') tends to turn clockwise. Couples in the same plane, or in parallel planes, are therefore distinguished as to their **sense**; and this sense is expressed by the algebraic sign attributed to the moment. Thus, the moment of the couple (F, p) in Fig. 30, is $+Fp$, that of the couple (F'', p'') is $-F''p''$.

129. *The effect of a couple is not changed by translation.*

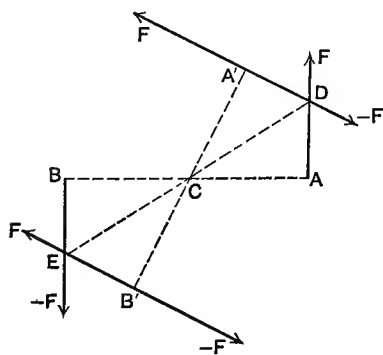
Let $AB=p$ (Fig. 31) be the arm of the couple (F, p) in its original position, and $A'B'$ the same arm in a new position parallel to the original one in the same plane, or in any parallel plane. By introducing at each end of the new arm $A'B'$ two opposite forces F , $-F$, each equal and parallel to the original forces F , the given system is not changed (Art. 80). But the

two equal and parallel forces F at A and B' form a resultant $2F$ at the middle point O of the diagonal AB' of the parallelogram $ABB'A'$. Similarly, the two forces $-F$ at B and A' are together equivalent to a resultant $-2F$ at the same point O . These two resultants, being equal and opposite and acting in the same line, are together equivalent to 0. Hence the whole system reduces to the force F at A' and the force $-F$ at B' , which form, therefore, a couple equivalent to the original couple at AB .



130. *The effect of a couple is not changed by rotation in its plane.*

Let AB (Fig. 32) be the arm of the couple in the original position, C its middle point, and let the couple be turned about C into the position $A'B'$. Applying again at A' , B' equal and opposite forces each equal to F , the forces $-F$ at A' and F at A will form a resultant acting along CD , while F at B' and $-F$ at B give an equal and opposite resultant along CE . These two



resultants destroy each other and leave nothing but the couple formed by F at A' and $-F$ at B' , which is therefore equivalent to the original couple.

Any other displacement of the couple in its plane, or to a parallel plane, can be effected by a translation combined with a rotation about the middle point of its arm in its plane.

The effect of a couple is therefore not changed by any displacement in its plane or to a parallel plane.

131. *The effect of a couple is not changed if its force F and its arm p be changed simultaneously in any way, provided their product Fp remain the same.*

Let $AB=p$ be the original arm (Fig. 33), F the original force of the couple; and let $A'B'=p'$ be the new arm. The introduction of two equal and opposite forces F' at A' , and also at B' , will not change the given system $F, -F$. Now, selecting for

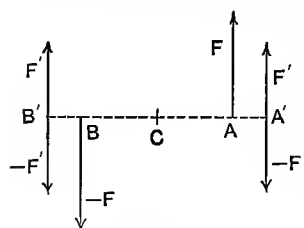


Fig. 33.

F' a magnitude such that $F'p'=Fp$, the force F at A and the force $-F'$ at A' combine (Arts. 104-106) to form a parallel resultant through C , the middle point of the arm, since for this point $F \cdot \frac{1}{2}p + (-F') \cdot \frac{1}{2}p' = 0$. Similarly, $-F$ at B and F' at B' give a resultant of the same magni-

tude, in the same line through C , but of opposite sense. These two resultants thus destroying each other, there remains only the couple formed by F' at A' and $-F'$ at B' , for which $Fp=F'p'$.

132. It results from the last three articles that the only essential characteristics of a couple are (a) the numerical value of the moment; (b) the sense, or direction of rotation; and (c) what has been called the "aspect" of its plane, *i.e.* the direction of any normal to this plane.

It is to be noticed that the plane of the two forces forming the couple is not an essential characteristic of the couple; just as the point of application of a force is not an essential characteristic of the force (see Art. 84).

Now the three characteristics enumerated above can all be indicated by a *vector* which can therefore serve as the geometrical representative of the couple. Thus, the couple formed by the forces $F, -F$ (Fig. 34), whose perpendicular distance is p , is represented by the vector $AB=Fp$ laid off on any normal to the plane of the couple. The sense is indicated by

drawing the vector toward that side of the plane from which the couple is seen to rotate counter-clockwise.

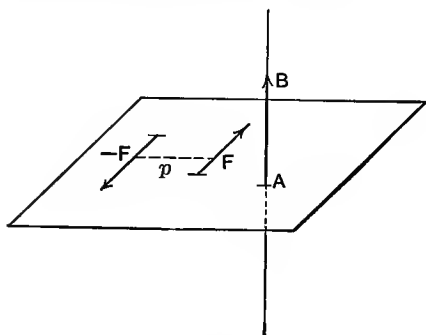


Fig. 34.

We shall call this geometrical representative AB of the couple simply the **vector** of the couple. It is sometimes called its *moment*, or its *axis*, or its *axial moment*.

133. As was pointed out in Art. 112, a couple is equivalent to a single force acting along a line at infinity. Couples are, therefore, used in statics to avoid the introduction of such forces whose line of action is at an infinite distance, just as in kinematics a rotation about an axis at infinity receives the special name of *translation*, and an angular velocity about an axis at infinity is called a *velocity of translation*.

It has been shown in *Kinematics*, Arts. 64, 65, that two equal and opposite rotations about parallel axes produce a translation, and in *Kinematics*, Art. 256, that two equal and opposite angular velocities about parallel axes produce a velocity of translation; similarly, two equal and opposite forces along parallel lines form a new kind of quantity called a *couple of forces*, or simply a *couple*.

While rotations, angular velocities, and forces are represented by *rotors*, *i.e.* by vectors confined to definite lines, translations, velocities of translation, and couples have for their geometrical representatives vectors not confined to particular lines.

Just as in the case of couples of infinitesimal rotations and of couples of angular velocities, the vector representing a couple

of forces has for its magnitude and sense those of the moment of the couple, and for its direction that perpendicular to the plane of the couple.

It is due to this analogy between the two fundamental conceptions that a certain dualism exists between the theories of statics and kinematics, so that a large portion of the theory of kinematics of a rigid body might be made directly available for statics by simply substituting for angular velocity and velocity of translation the corresponding ideas of force and couple.

134. When any number of couples act on a rigid body their resultant can readily be found. Representing each couple by its vector, we have only to combine these vectors by geometrical addition. In the particular case when the couples all lie in parallel planes, or in the same plane, their vectors may be taken in the same line, and add, therefore, algebraically.

Hence, *the resultant of any number of couples is a single couple whose vector is the geometric sum of the vectors of the given couples.*

Conversely, a couple can be resolved into components by resolving its vector into components.

135. To combine a single force P with a couple (F, p) lying

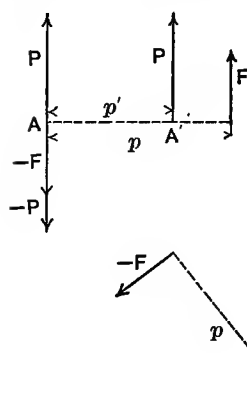


Fig. 35.

in the same plane it is only necessary to place the couple in its plane into such a position (Fig. 35) that one of its forces, say $-F$, shall lie in the same line and in opposite sense with the single force P , and to transform the couple (F, p) into a couple (P, p') , by Art. 131, so that $Fp = Pp'$. The original single force P and the force $-P$ of the transformed couple destroying each other at A , there remains only

the other force P , at A' , of the transformed couple which is par-

allel and equal to the original single force P , and has the distance

$$p' = \frac{F}{P} p$$

from it.

Hence, *a couple and a single force in the same plane are together equivalent to a single force equal and parallel to, and of the same sense with, the given force, but at a distance from it which is found by dividing the moment of the couple by the single force.*

136. Conversely, *a single force P applied at a point A of a rigid body can always be replaced by an equal and parallel force P of the same sense, applied at any other point A' of the same body, in connection with the couple formed by P at A and $-P$ at A' .*

137. The proposition of Art. 135 applies even when the force lies in a plane parallel to that of the couple, since the couple can be transferred to any parallel plane without changing its effect.

If the single force intersects the plane of the couple, it can be resolved into two components, one lying in the plane of the couple, while the other is at right angles to this plane. On the former component the couple has, according to Art. 135, the effect of transferring it to a parallel line. We thus obtain *two non-intersecting, or skew, forces at right angles to each other.*

Let P be the given force, and let it make the angle α with the plane of the given couple, whose force is F and whose arm is p . Then $P \sin \alpha$ is the component at right angles to the plane of the couple, while $P \cos \alpha$ combines with the couple whose moment is Fp to a force $P \cos \alpha$ in the plane of the couple; this force $P \cos \alpha$ is parallel to the projection of P on the plane, and has the distance $\frac{Fp}{P \cos \alpha}$ from this projection.

Hence, in the most general case, *the combination of a single force and a couple can be replaced by the combination of two single forces crossing each other at right angles; it can be reduced to a single force only when the force is parallel to the plane of the couple.*

138. Exercises.

(1) Show that the moment of a couple can be represented by the area of the parallelogram formed by the two forces of the couple, or by the area of the triangle formed by joining any point on the line of one of the forces to the ends of the other force.

(2) Show that the sum of the moments of two forces forming a couple is the same for any point in the plane of the couple.

(3) Show, by means of Arts. 129–131, how to combine any number of couples situated in the same plane, or in parallel planes.

(4) Find the resultant of two couples situated in non-parallel planes, without using the vectors of the couples.

V. *Plane Statics.*

I. THE CONDITIONS OF EQUILIBRIUM.

139. Suppose a rigid body to be acted upon by any number of forces, all of which are situated in the same plane. To reduce such a *plane system of forces* to its simplest form the proposition of Art. 136 may be used. This proposition allows us to transfer all the forces to a common origin, by introducing, in addition to each force, a certain couple in the same plane. The concurrent forces can then be combined into their resultant by geometric addition, or by forming their force polygon (Art. 98); and the couples lying all in the same plane combine by algebraic addition of their moments into a resultant couple (Art. 134).

Thus, let F (Fig. 36) be one of the forces of the given plane system, P its point of application. Selecting any point O in the plane as origin, apply at

O two equal and opposite forces F , $-F$, each equal and parallel to the given force F ; and let p be the perpendicular distance of the origin O from the line of action of the given force F . The force F at P is

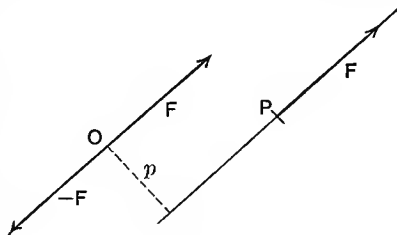


Fig. 36.

equivalent to the force F at O in connection with the couple formed by F at P and $-F$ at O ; the moment of this couple is Fp , its vector is perpendicular to the plane of the system.

Proceeding in the same way with every force of the given system, all forces are transferred to the common origin O . The whole system is therefore equivalent to their resultant R passing through O , in connection with the resulting couple $H = \sum Fp$.

140. The given system of forces will be in equilibrium if the following two **conditions of equilibrium** are fulfilled :

$$R=0, \quad H=0.$$

It will be noticed that the moment Fp of the couple introduced by transferring the force F to the point O is the moment of the force F with respect to this point O .

Hence, *a plane system of forces is in equilibrium if (a) its resultant is zero, and (b) the algebraic sum of the moments of all its forces is zero with respect to any point in its plane.*

141. It is evident that the magnitude and direction of the resultant R do not depend on the selection of the origin O . But the position of this resultant and the magnitude of the resulting couple H will in general differ for different points selected as origin. Indeed, the origin can be so taken as to make the couple H vanish (unless the resultant R be zero); that is, the whole system can be reduced to a single resultant.

To do this (see Art. 135), it is only necessary, after determining R and H for some point O , to transfer R to a parallel line at such a distance r from its original position as to make the moment Rr of the couple introduced by the transfer equal and opposite to the moment ΣFp ; *i.e.* we must take (Art. 135)

$$r = -\frac{H}{R}.$$

The line along which this single resultant acts is called the **central axis** of the given system of forces.

142. For a purely analytical reduction of a plane system of forces the system is referred to rectangular axes Ox , Oy , arbitrarily assumed in the plane (Fig. 37). Every force F is resolved at its point of application $P(x, y)$ into two components X , Y , parallel to the axes, so that

$$X = F \cos \alpha, \quad Y = F \sin \alpha,$$

α being the angle made by F with the axis Ox . At the origin O two equal and opposite forces $X, -X$ are applied along Ox , and two equal and opposite forces $Y, -Y$ along Oy . Thus, X at P is equivalent to X at O in combination with the couple formed by X at P and $-X$ at O ; the moment of this couple is evidently $-yX$. Similarly, Y at P is replaced by Y at O in combination with a couple whose moment is xY .

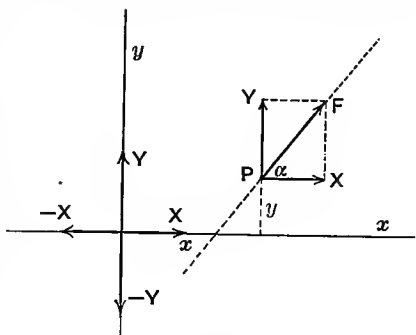


Fig. 37.

The force F at P is therefore equivalent to the two forces X, Y at O in combination with a couple whose moment is $xY - yX$.

Proceeding in the same way with every given force, we obtain a number of forces X along Ox which can be added algebraically into ΣX , and a number of forces Y along Oy which give ΣY . These two rectangular forces form the resultant

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$$

whose direction is given by

$$\tan \alpha = \frac{\Sigma Y}{\Sigma X},$$

where α is the angle between Ox and R .

In addition to this, we obtain a number of couples $xY - yX$ whose algebraic sum forms the resulting couple

$$H = \Sigma (xY - yX).$$

143. The whole system is thus found equivalent to a resultant force R in combination with a resultant couple H in the same plane with R . The *conditions of equilibrium* $R=0$, $H=0$ (Art. 140) can therefore be expressed analytically by the three equations

$$\Sigma X=0, \quad \Sigma Y=0, \quad \Sigma (xY - yX)=0.$$

144. If R be not zero, R and H can be combined into a single resultant R' equal and parallel to R at the distance $-H/R$ from it (see Art 141). The equation of the line of this single resultant R' , *i.e.* the central axis of the system of forces, is found by considering that it makes the angle α with the axis of x and that its distance from the origin is

$$H/R = \Sigma (xY - yX) / \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}.$$

Hence its equation is

$$\xi \cdot \Sigma Y - \eta \cdot \Sigma X - \Sigma (xY - yX) = 0.$$

If $R=0$, the system is equivalent to the couple

$$H = \Sigma (xY - yX),$$

unless H itself be also zero, in which case the system is in equilibrium.

145. The same results can be obtained by a transformation of co-ordinates. Let $R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$ and $H = \Sigma (xY - yX)$ be the resultant force and couple for a point O as origin. If some other point O' , whose co-ordinates with respect to O are ξ, η , be taken as new origin and x', y' be the co-ordinates of the point of application P of the force F for parallel axes through O' , the resultant R remains the same while the resulting couple becomes

$$\begin{aligned} H' &= \Sigma (x'Y - y'X) = \Sigma [(x - \xi)Y - (y - \eta)X] \\ &= H - \xi \Sigma Y + \eta \Sigma X. \end{aligned}$$

Hence this new couple will vanish whenever the origin $O'(\xi, \eta)$ is taken on the straight line whose equation referred to the original axes is

$$\Sigma Y \cdot \xi - \Sigma X \cdot \eta - H = 0.$$

This equation of the central axis agrees with the equation found in Art. 144; it represents the line of action of the single resultant to which the system can be reduced.

146. The following examples will illustrate the application of the conditions of equilibrium. To establish these conditions in any particular problem it will generally be found best to resolve the forces along two rectangular directions and equate the sums of the components to zero; and then to "take moments," *i.e.* equate to zero the sum of the moments of all the forces with respect to some point conveniently selected as origin.

147. A homogeneous straight rod $AB=2l$ (Fig. 38) of weight W rests with one end A on a smooth horizontal plane AH , and with the point E ($AE=e$) on a cylindrical support, the axis of the cylinder being at right angles to the vertical plane containing the rod. Determine what horizontal force F must be applied at a given point F of the rod ($AF=f>e$) to keep the rod in equilibrium when inclined to the horizon at an angle θ .

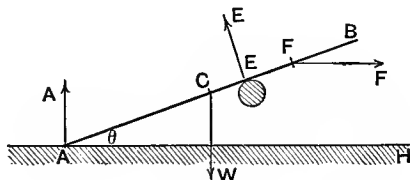


Fig. 38.

The rod exerts a certain unknown pressure on each of the supports at A and E , in the direction of the normals to the surfaces of contact, provided there be no friction, as is here assumed. The supports may therefore be imagined removed if forces A , E , equal and opposite to these pressures, be introduced; these forces A , E are called the *reactions* of the supports. The rod itself is here regarded as a straight line; its weight W is applied at its middle point C .

Taking A as origin and AH as axis of x , the resolution of the forces gives

$$\Sigma X \equiv F - E \sin \theta = 0, \quad (1)$$

$$\Sigma Y \equiv A - W + E \cos \theta = 0. \quad (2)$$

Taking moments about A , we find

$$E \cdot e - W \cdot l \cos \theta - F \cdot f \sin \theta = 0. \quad (3)$$

Eliminating F from (1) and (2), we have

$$E = \frac{l \cos \theta}{e - f \sin^2 \theta} W;$$

hence from (2),

$$A = \left(1 - \frac{l \cos^2 \theta}{e - f \sin^2 \theta} \right) W$$

and finally from (1),

$$F = \frac{l \sin \theta \cos \theta}{e - f \sin^2 \theta} W.$$

148. A cylinder of length $2l$ and radius r rests with the point A of the circumference of its lower base on a horizontal plane and with the point B of the circumference of its upper base against a vertical wall. The vertical plane through the axis of the cylinder contains the points A , B , and is perpendicular to the intersection of the vertical wall and the horizontal plane. If there be no friction at A and B , what horizontal force F applied at A will keep the cylinder in equilibrium? When is this force $F = 0$?

Let G be the centre of gravity of the cylinder; W its weight; A , B the reactions at A , B ; and θ the given angle between AB and the horizontal plane. Then $B - F = 0$, $A - W = 0$, and taking moments about A ,

$$W(l \cos \theta - r \sin \theta) = B \cdot 2l \sin \theta.$$

Hence $A = W$,

$$B = F = W \cdot \frac{l \cos \theta - r \sin \theta}{2l \sin \theta}$$

$$= \frac{1}{2} \left(\cot \theta - \frac{r}{l} \right) \cdot W.$$

If either the dimensions of the cylinder, or the angle θ , be such as to make $\tan \theta = l/r$, no force F will be required to maintain equilibrium; G and A will then lie in the same vertical line.

149. The homogeneous rod $AB = 2l$ of weight W is jointed at A , so as to turn about A in a vertical plane. A string BC attached to the end B of the rod runs at C over a smooth pulley, and carries a weight P . The axis of the pulley C is parallel to, and in the same vertical plane

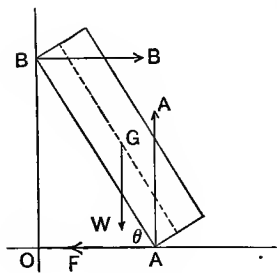


Fig. 39.

with, the axis of the joint A; $AC = h$. Find the position of equilibrium and the pressure on the axis of the joint A. (Fig. 40.)

To reduce to a purely statical problem, cut the string between B and C and introduce the tension, which is $= P$; also, replace the pressure A by its horizontal and vertical components A_x, A_y . Then, if $\angle ACB = \phi$, $\angle BAC = \theta$, the conditions of equilibrium give

$$A_x = P \sin \phi, \quad A_y = W - P \cos \phi,$$

$$P \cdot h \sin \phi = W \cdot l \sin \theta.$$

From the last equation,

$$\frac{\sin \phi}{\sin \theta} = \frac{l}{h} \cdot \frac{W}{P},$$

while from the triangle ABC,

$$\frac{\sin \phi}{\sin \theta} = \frac{2l}{BC};$$

hence $BC = 2hP/W$, i.e. if we take h to represent W , P will be represented by $\frac{1}{2} BC$.

For the total pressure A we have

$$A^2 = A_x^2 + A_y^2 = W^2 + P^2 - 2WP \cos \phi,$$

i.e. A is the third side of a triangle having W and P for the two other sides, and ϕ for the included angle. The magnitude of A is therefore represented by the median from A in the triangle ABC on the same scale on which W is represented by h . But this median gives also the direction of A ; for we have

$$\frac{A_y}{A_x} = \frac{W - P \cos \phi}{P \sin \phi} = \frac{h - \frac{BC}{2} \cos \phi}{\frac{BC}{2} \sin \phi}.$$

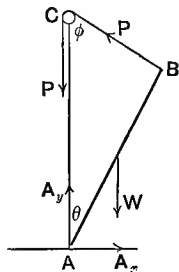


Fig. 40.

150. A weightless rod AB rests without friction on two planes inclined to the horizon at angles α, β , and carries a weight W at the point D. The intersection (C) of these planes is horizontal and at right angles to the vertical plane through AB. Find the inclination θ of AB to the horizon, and the pressures at A and B. (Fig. 41.)

As there are only three forces, viz. the weight W and the reactions A and B , their lines must intersect in a point E . Resolving horizontally and vertically, we have

$$A \sin \alpha = B \sin \beta,$$

$$A \cos \alpha + B \cos \beta = W,$$

$$\text{whence } A = \frac{\sin \beta}{\sin(\alpha + \beta)} W,$$

$$B = \frac{\sin \alpha}{\sin(\alpha + \beta)} W.$$

Taking moments about D , we find with $AD = a$, $DB = b$,

$$A \cdot a \sin DAE = B \cdot b \sin DBE,$$

$$\text{or } Aa \cos(\alpha + \theta) = Bb \cos(\beta - \theta);$$

to eliminate A and B , divide by the first equation above :

$$a \frac{\cos(\alpha + \theta)}{\sin \alpha} = b \frac{\cos(\beta - \theta)}{\sin \beta};$$

solving for θ , we finally obtain

$$\tan \theta = \frac{a \cot \alpha - b \cot \beta}{a + b}.$$

151. Exercises.

(1) A homogeneous rod $AB = 2l = 8$ ft., weighing $W = 20$ lbs., rests with one end A on a horizontal plane AH , and with the point E on a support whose height above AH is $DE = h = 3$ ft. A horizontal cord $AD = d = 4$ ft. holds the rod in equilibrium. Find the tension T of this cord, and the reactions at A and E .

(2) A weightless rod AB of length l can turn freely about one end A in a vertical plane. A weight W is suspended from a point C of the rod; $AC = c$. A string BD attached to the end B of the rod holds it in equilibrium in a horizontal position, the angle ABD being $\alpha = 150^\circ$. Find the tension T of the string and the resulting pressure A on the hinge at A .

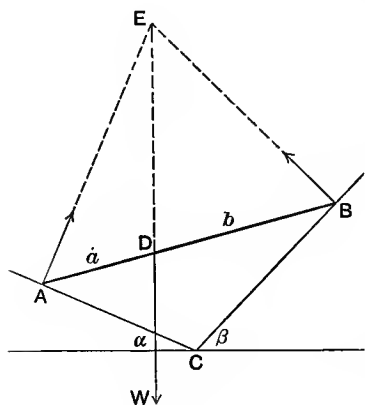


Fig. 41.

(3) A uniform rod $AB = 2l$ of weight W rests with its upper end A against a smooth vertical wall, while its lower end B is fastened by a string of given length, $BC = 2b$, to a point C in the wall. The rod and the string are in the vertical plane at right angles to the wall. Find the position of equilibrium, *i.e.* the angle $\phi = ACB$, the tension T of the string, and the pressure A against the wall.

(4) A uniform rod $AB = 2l$ of weight W rests with one end A on a smooth horizontal plane AC , with the other end B against a smooth vertical wall BC , the vertical plane through AB being at right angles to the intersection C of the wall with the horizontal plane. The rod is kept in equilibrium by a string EC . Find the tension T of this string if the angles $CAB = \theta$ and $ECA = \phi$ are given.

(5) A weightless rod $AB = l$ can revolve in a vertical plane about a hinge at A ; its other end B leans against a smooth vertical wall whose distance from A is $AD = a$. At the distance $AC = c$ from A , a weight W is suspended. Find the horizontal thrust A_x at A and the normal pressures A_y and B at A and B .

(6) The same as (5) except that at B the rod rests on a smooth horizontal cylinder whose axis is at right angles to the vertical plane through AB . In which of the two problems is the horizontal thrust A_x at A least?

(7) The lower end A of a smooth uniform rod AB of weight W rests on a smooth horizontal plane making an angle θ with it. At the point C it rests on a smooth cylinder whose axis is horizontal and at right angles to the vertical plane through the rod; at D the rod is pressed upon by another smooth cylinder whose axis is parallel to that of the cylinder at C . Determine the reactions at A , C , D , if W , θ , $AB = 2l$, $CD = a$ are given.

(8) A smooth weightless rod $AB = l$ rests at C on a smooth horizontal cylinder whose axis is at right angles to the vertical plane through the rod; its lower end A leans against a smooth vertical wall whose distance from C is $CD = a$; from its upper end B a weight W is suspended. Determine the distance $AC = x$ for equilibrium, and the reactions at A and C .

(9) A uniform rod of weight W is hinged at its lower end A , while its upper end B leans against a smooth vertical wall. The rod is inclined at an angle θ to the vertical, and carries three weights, each equal to w , at three points dividing the rod into four equal parts. Determine the pressure on the wall and the reaction of the hinge.

(10) A homogeneous rod $AB = 2l$ of weight W rests with one end A on the inside of a fixed hemispherical bowl of diameter $2a$ and leans at C on the horizontal rim of the bowl, so that the other end B is outside. Determine the inclination to the horizon θ in the position of equilibrium.

2. STABILITY.

152. The equilibrium of the forces acting on a rigid body may subsist while the body is in motion. Thus, if the motion consist in a mere translation with constant velocity, the equilibrium will not be disturbed during the motion if the forces remain equal and parallel to themselves.

If, however, the body be subjected to a rotation, this will in general not be the case. The present considerations are restricted to the case of plane motion; the forces are supposed to lie in the plane of the motion and to remain equal and parallel to themselves and applied at the same points of the body.

153. Let A_1A_2 (Figs. 42 and 43) be a rigid rod having two equal and opposite forces F_1, F_2 applied at its extremities in the

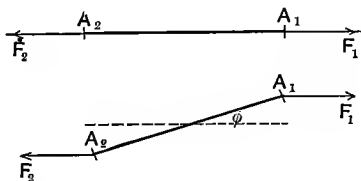


Fig. 42.

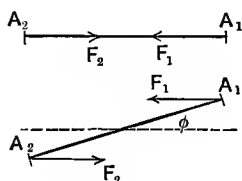


Fig. 43.

direction of the line A_1A_2 . Let this rod be turned by an angle ϕ about an axis at right angles to A_1A_2 . In the new position the forces F_1, F_2 , instead of being in equilibrium, form a couple whose moment is $\pm F_1 \cdot A_1A_2 \sin \phi$.

If in the original position of the rod the forces tend to increase the distance A_1A_2 (Fig. 42), the couple in the new position will

tend to bring the rod back to its position of equilibrium. In this case the original position of the rod is said to be a position of **stable equilibrium**. The effect of the earth's magnetism on the needle of a compass offers a familiar example.

If, however, in the original position the forces tend to diminish the distance A_1A_2 (Fig. 43), the couple arising after displacement tends to increase the displacement and thus to remove the rod still farther from equilibrium. The original position in this case is said to be one of **unstable equilibrium**. The weight resting on a vertical post and the reaction of the support on which the rod stands may be taken as an illustration.

Finally, a third case would arise if the forces F_1, F_2 , being still equal and opposite, were applied at one and the same point of the rod. The forces would then remain in equilibrium after any displacement of the rod; such equilibrium is called **neutral** or **astatic**.

154. These different cases of equilibrium can be distinguished by the algebraic sign of the product $A_1A_2 \cdot F_1 = A_2A_1 \cdot F_2$, which is negative for stable equilibrium, since A_1A_2 and F_1 have opposite sense (Fig. 42), positive for unstable equilibrium (Fig. 43), and indeterminate (since $A_1A_2=0$) for neutral equilibrium.

It is to be noticed that these considerations will hold whether the rotation of angle ϕ take place in the positive or negative sense. But they hold only within certain limits for the angle of rotation. Thus, in the example illustrated by Figs. 42 and 43, when ϕ reaches the value π , the nature of the equilibrium is changed.

155. Strictly speaking, investigations of stability are not purely statical, but require a kinetic examination of the subsequent motion. However, the principles of statics are sufficient to determine the nature of the equilibrium for infinitesimal displacements, *i.e.* when only the initial motion of the body is considered.

The theory of astatic equilibrium forms a special branch of mechanics called **astatics**; its object is to determine the conditions under which a system of forces acting on a rigid body remains in equilibrium when the body is subjected to any displacement while the forces remain applied at the same points of the body and retain their magnitude, direction, and sense.

156. The equilibrium of forces acting on one and the same point is evidently always astatic.

In the case of a plane system of forces acting on a plane figure in its plane, the only displacement that need be considered is a rotation about an axis at right angles to the plane. For every displacement of a plane figure in its plane can be reduced to a rotation about a certain centre in the plane.

Instead of turning the body or plane figure by an angle ϕ , we may turn all the forces about their points of application by the same angle in the opposite sense.

157. If the plane system consists of two forces in equilibrium, they must be equal and opposite, and act in the same line; this case has been considered in Art. 154.

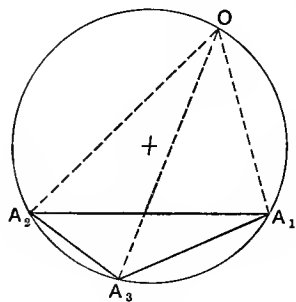


Fig. 44.

If there be three forces F_1, F_2, F_3 in the same plane in equilibrium, applied at the points A_1, A_2, A_3 , they must meet in a point O , and fulfil the parallelogram law.

After turning each force about its point of application by the same angle, the forces will, in general, cease to intersect in a point, and hence to be in equilibrium. If, however, the original meeting point O of the forces be situated on the circle described through A_1, A_2, A_3 (Fig. 44), the forces will continue to intersect at some point of this circle when turned through some angle, because the angles between the forces remain constant.

Hence, *three forces* F_1, F_2, F_3 *in the same plane, applied at points* A_1, A_2, A_3 , *are in astatic equilibrium if they meet in a point* O *situated on the circle passing through* A_1, A_2, A_3 .

The condition of equilibrium of the three forces also requires that

$$\frac{F_1}{\sin(F_2F_3)} = \frac{F_2}{\sin(F_3F_1)} = \frac{F_3}{\sin(F_1F_2)};$$

by the property of the circle (Fig. 44), we have $\angle(F_2F_3) = A_1$, $\angle(F_3F_1) = A_2$, $\angle(F_1F_2) = \pi - A_3$; and as the sines of these angles of the triangle $A_1A_2A_3$ are proportional to the opposite sides, we have

$$\frac{F_1}{A_2A_3} = \frac{F_2}{A_3A_1} = \frac{F_3}{A_1A_2};$$

i.e. three forces in astatic equilibrium are to each other as the sides of the triangle formed by their points of application.

158. The results of the preceding article can be interpreted from a somewhat different point of view. Let two of the forces, F_1 and F_2 , be given, and let it be required to determine their resultant for astatic equilibrium. This resultant F_3 must evidently pass through a definite point A_3 of the circle described through the points of application A_1, A_2 of the given forces and their intersection O . This point A_3 , through which the resultant must pass, howsoever the two given forces be turned about A_1, A_2 , is called the **centre of the forces**.

If the two given forces be parallel, the point O lies at infinity, and the circle through A_1, A_2, O becomes the straight line A_1A_2 . The point A_3 is therefore situated on this line and divides the distance A_1A_2 in the inverse ratio of the forces F_1, F_2 , by Art. 157. Compare Art. 110.

159. These results are readily generalised. Any plane system of forces has a centre unless the resultant be zero. To find the centre we have only to combine the forces in succession, *i.e.* to find the centre of two of the forces, then the centre of their resultant and a third force, etc.

It has been shown in Arts. 141 and 144 that a plane system of forces whose resultant does not vanish can always be reduced to a single resultant R whose line is called the *central axis* of the system. It appears now that if the forces be all turned by the same angle θ about their points of application, the line of the resultant, or the central axis, will turn about a certain fixed point called the *centre* of the system. For a system of parallel forces the existence of such a centre has already been proved in Art. 110.

160. Analytically, the centre of a plane system of forces is found as the intersection of the two positions of the central axis before and after any displacement of the plane figure, or body, on which the forces act.

By Arts. 144 and 145 the equation of the central axis is

$$\Sigma Y \cdot \xi - \Sigma x \cdot \eta - \Sigma (xY - yX) = 0. \quad (1)$$

Let the figure with the axes of co-ordinates be turned through an angle ϕ about an axis through the origin perpendicular to its plane, while the forces keep their original directions. The central axis of the forces in the new position will have an equation of the same form as before, in which, however, x, y, ξ, η are referred to the new system of co-ordinates. To find the equation of the central axis in the old co-ordinates, we have to substitute $x \cos \phi - y \sin \phi$ for x , $x \sin \phi + y \cos \phi$ for y , and similarly for ξ, η . This gives

$$\begin{aligned} & \Sigma Y \cdot (\xi \cos \phi - \eta \sin \phi) - \Sigma X \cdot (\xi \sin \phi + \eta \cos \phi) \\ & - \Sigma [(x \cos \phi - y \sin \phi) Y - (x \sin \phi + y \cos \phi) X] = 0, \end{aligned}$$

or collecting the terms containing $\cos \phi$ and $\sin \phi$, respectively,

$$\begin{aligned} & [\Sigma Y \cdot \xi - \Sigma X \cdot \eta - \Sigma (xY - yX)] \cos \phi \\ & - [\Sigma x \cdot \xi + \Sigma Y \cdot \eta - \Sigma (xX + yY)] \sin \phi = 0. \quad (2) \end{aligned}$$

The centre being the intersection of the lines (1) and (2), its

co-ordinates are found by solving these equation for ξ and η , or since the coefficient of $\cos \phi$ in (2) vanishes by (1), by solving the equations

$$\Sigma Y \cdot \xi - \Sigma X \cdot \eta = \Sigma (xY - yX), \quad (3)$$

$$\Sigma x \cdot \xi + \Sigma Y \cdot \eta = \Sigma (xX + yY). \quad (4)$$

Putting, for shortness, $\sqrt{(\Sigma X)^2 + (\Sigma Y)^2} = R$, $\Sigma (xY - yX) = H$, $\Sigma (xX + yY) = K$, we find the co-ordinates of the centre,

$$\xi = \frac{\Sigma Y \cdot H + \Sigma X \cdot K}{R^2}, \quad \eta = \frac{\Sigma Y \cdot K - \Sigma X \cdot H}{R^2}. \quad (5)$$

161. By the rotation of the figure, the magnitude of the resultant R of the system is of course not changed. But the resulting couple H for the origin, or what amounts to the same, the moment of the system about the origin, is changed and becomes, by Art. 160,

$$\begin{aligned} H' &= \Sigma [(x \cos \phi - y \sin \phi) Y - (x \sin \phi + y \cos \phi) X] \\ &= \Sigma (xY - yX) \cdot \cos \phi - \Sigma (xX + yY) \cdot \sin \phi \\ &= H \cos \phi - K \sin \phi. \end{aligned} \quad (6)$$

This couple H' vanishes if the figure be turned through an angle ϕ determined by the equation

$$\tan \phi = \frac{H}{K}. \quad (7)$$

162. If the system of forces be originally in equilibrium, we have $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma (xY - yX) = 0$ (Art. 143). Hence after turning the figure through an angle ϕ , the forces will be equivalent to the couple

$$H' = -K \sin \phi. \quad (8)$$

This couple has its greatest value when $\phi = \pi/2$; it vanishes only when $\phi = \pi$, in which case the system will again be in equilibrium.

163. The stability of a plane system in equilibrium depends on the algebraic sign of the quantity,

$$K = \Sigma (xX + yY), \quad (9)$$

which can therefore be called the **stability function**. If this function be positive, the equilibrium is stable; if it be negative, the equilibrium is unstable; finally, if $K=0$, the system is astatic, and the equilibrium is neutral.

The proof follows at once from equation (8). This equation shows that, for a positive K , the moment of the couple to which the system becomes equivalent when the figure is turned through an angle ϕ has a sign opposite to that of the angle ϕ ; hence this couple will tend to turn the body back into the position of equilibrium. Similarly, if K be negative, H' agrees in sign with ϕ and tends therefore to increase this angle.

164. Exercises.

(1) Explain the nature of the equilibrium of a body of weight W supported at a single point according to the position of that point above the centroid G , below G , and at G (*common balance*).

(2) A homogeneous rod $AB = 2l$ of weight W leans with the lower end A against a vertical wall and rests with the point C ($AC = c > l$) on a cylindrical support. Show that the equilibrium is unstable.

(3) A body of weight W is placed on a horizontal plane. Show that the equilibrium is stable if W meets the horizontal plane at a point A within the area of contact and that it is unstable if A lies on the contour of this area. If the actual area of contact have re-entrant angles, or consist of several detached portions, the area bounded by a thread drawn tightly around the actual area, or areas, of contact must be substituted.

(4) An oblique cylinder rests with its circular base on a horizontal plane in unstable equilibrium. If the length of its axis be twice the diameter of its base, what is the inclination of the axis to the horizon?

(5) Show how to determine graphically the stability of a retaining wall against toppling over the front edge of the base, the pressure of the earth behind the wall being given in magnitude, direction, and position.

3. JOINTED FRAMES.

165. The equations of equilibrium are derived on the suppositions that all the forces of the given system act on one and the same rigid body and that this body is perfectly free to move. Hence, in applying these equations to determine the equilibrium of an engineering structure, a machine, etc., each rigid body must be considered separately, and the reactions required to make the body free must be introduced. It will be shown in a subsequent section how the principle of work makes it possible to dispense with some of these precautions.

When two rigid rods are connected by a pin-joint whose axis is perpendicular to the plane of the rods, the action of either rod on the other at the joint is represented by a single force whose direction is in general unknown. Sometimes considerations of symmetry will allow to determine this direction.

If a rigid rod, in equilibrium, be hinged at both ends and not acted upon by any other forces, the reactions of the hinges must of course be along the rod, and must be equal and opposite.

166. *Two rods AC, BC (Fig. 45) in a vertical plane, hinged together at C, rest with the ends A, B on a horizontal plane, and carry a weight W suspended from the joint C. If the proper weight of the rods be neglected, determine the normal pressures A_y , B_y and the horizontal thrusts A_x , B_x at A, B.*

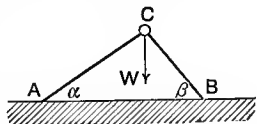


Fig. 45.

Resolving the weight W along CA , CB into W_A , W_B and considering the rod AC alone it appears that the total reaction at A is along AC and $= W_A$; hence resolving W_A in the horizontal and vertical directions, A_x and A_y are found; similarly for BC . If α , β be the angles at A and B in the triangle ABC , we find

$$W_A = \frac{\cos \beta}{\sin(\alpha + \beta)} W, \quad W_B = \frac{\cos \alpha}{\sin(\alpha + \beta)} W;$$

$$A_x = B_x = \frac{\cos \alpha \cos \beta}{\sin(\alpha + \beta)} W, \quad A_y = \frac{\sin \alpha \cos \beta}{\sin(\alpha + \beta)} W, \quad B_y = \frac{\cos \alpha \sin \beta}{\sin(\alpha + \beta)} W.$$

As the horizontal thrusts at A and B are equal, it makes no difference whether the rods be hinged to the support at A and B , or whether the thrust is taken up by lateral supports, or by a string connecting the ends A, B of the rods.

167. Two equal homogeneous rods AC, BC (Fig. 46) are hinged at A, B, C so as to form a triangle whose height h is vertical and whose base $AB = 2b$ is horizontal. The weight of each rod being W , find the reactions at the joints.

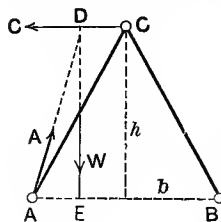


Fig. 46.

Owing to the symmetry of the figure, the reactions at C must be equal and opposite and horizontal. The rod AC is subject to three forces only, viz. the horizontal reaction C , the weight W , and the reaction A ; the latter must therefore pass through the intersection D of C and W .

If the direction of W intersect AB at E and the scale of forces be taken so as to have W represented by $DE = h$, DEA will be the force polygon; hence EA represents C and AD represents A on the same scale on which W is represented by h .

Analytically, the reactions are found by resolving the forces horizontally and vertically and taking moments about A :

$$A_x = C, \quad A_y = W, \quad C \cdot h = W \cdot \frac{b}{2};$$

whence $C = mW, \quad A = \sqrt{A_x^2 + A_y^2} = \sqrt{m^2 + 1} \cdot W,$

where $m = \frac{b}{2h}.$

168. Two equal homogeneous rods AC, BC , each of weight W , are hinged at C ; their ends A, B rest on a smooth horizontal plane; a third rod DE is hinged to them, connecting their middle points (Fig. 47).

The plane AB being smooth, the reaction at A is vertical; the reaction at C is horizontal owing to the symmetry; that at D is likewise horizontal if the weight of the rod DE be neglected, for then this rod is subject only to the reactions at its ends.

Resolving horizontally and vertically and taking moments about E , we find in this case

$$A = W, \quad C = D = W \cot \alpha,$$

where $\alpha = \angle BAC$.

If, however, the weight w of the rod DE cannot be neglected, we have at D a horizontal reaction D_x and a vertical reaction D_y . The equilibrium of DE requires that $2D_y = w$. Hence resolving and taking moments as before, we find

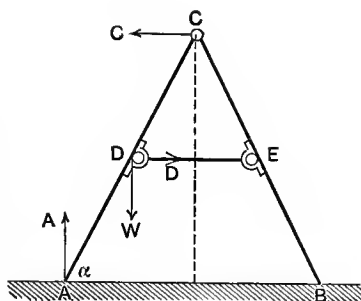


Fig. 47.

$$A = W - \frac{1}{2}w, \quad C = D_x = (W - \frac{1}{2}w) \cot \alpha, \quad D_y = \frac{1}{2}w.$$

169. Exercise.

(1) *Two homogeneous rods AC, BC of equal weight, but unequal length, are hinged together at C while their other ends are attached to fixed hinges A, B in the same vertical line. Show that the line of action of the reaction at C bisects AB.*

170. A triangular frame formed of rigid rods is rigid as a whole, even when the connections are pin-joints. A quadrangular frame with pin-joints becomes rigid only by the insertion of a diagonal.

The iron and steel trusses used for roofs and bridges generally consist of a system of triangles, or quadrangles with diagonals, so that the whole truss can be regarded as one rigid body, at least in first approximation.

Any one rod, or *member*, of the frame-work is thus acted upon by two equal and opposite forces, *i.e.* by a *stress*, in the direction of its length, the external forces, including the proper weight, being regarded as applied at the joints only. If the stress be a *tension*, *i.e.* if the forces tend to stretch or elongate the member, the latter is called a **tie**; a member subject to *compression* or crushing is called a **strut**.

171. For the purpose of dimensioning the members, it is necessary to know the stress in every member. The following

example illustrates a simple method for finding these stresses when the external forces are given.

Let the frame-work represented in Fig. 48 be cut in two along any line $\alpha\beta$; the portion on either side of this line must be in equilibrium under the action of its external forces and the

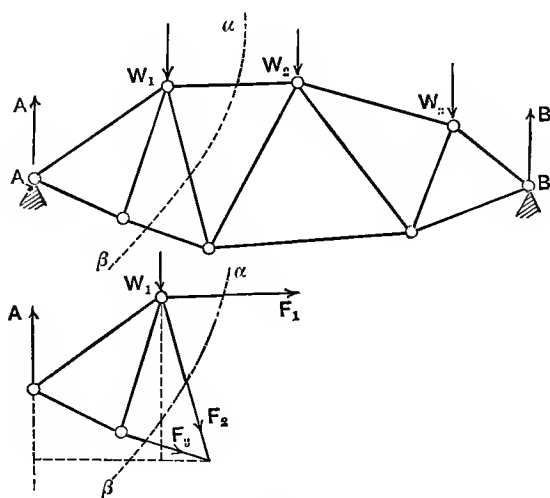


Fig. 48.

stresses in the members intersected by $\alpha\beta$. Thus, in the figure, the forces A , W_1 , F_1 , F_2 , F_3 form a system in equilibrium; hence, the sum of the moments of these forces with respect to any point must vanish.

To determine F_1 , take moments about the intersection of F_2 and F_3 ; thus F_2 and F_3 are eliminated from the equation of moments, and F_1 is found. Similarly F_2 is obtained by taking moments about the intersection of F_3 and F_1 . The arms of the moments are best taken from a correctly drawn diagram of the frame-work.

If only two members be intersected by $\alpha\beta$, the origin for the moments is taken first on one, then on the other, of the two members intersected.

By beginning at one of the supports and taking sections

through the successive panels, it will in the more simple cases be possible to draw the line $\alpha\beta$ so as to intersect not more than three members whose stresses are unknown. Thus the stresses in all the members can be determined.

172. Exercises.

(1) Find the stresses in the braced beam AB (Fig. 49), carrying a weight of 5 tons at each joint of the upper chord. The horizontal width of the panels is 10 ft., the middle vertical is 8 ft.

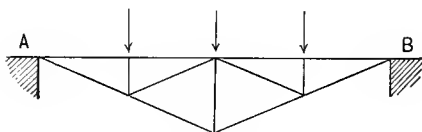


Fig. 49.

(2) In Fig. 50, the dimensions are in feet, the loads in tons. After the first panel the sections cannot be so taken as to intersect not more than three unknown stresses. But the girder can be regarded as obtained by the superposition of two girders (each carrying half the load), in one of which the diagonals CF , EH are wanting, while in the other DE , FG are wanting. Each of these can readily be computed.

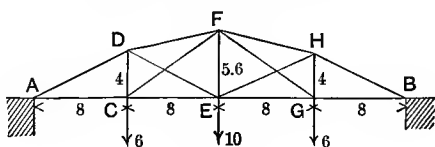


Fig. 50.

4. GRAPHICAL METHODS.

173. The graphical method explained in Art. 108 for determining the resultant of a system of parallel forces can be extended without difficulty to the general case of a plane system of forces. The only difference will appear in the form of the force polygon, which for parallel forces collapses into a straight line, while in the general case it is an ordinary (unclosed) polygon whose closing line represents the resultant

in magnitude and direction. In other words, when the forces are not parallel, they must be added geometrically, and not algebraically.

The construction of the funicular polygon and its properties are the same as for parallel forces.

If the force polygon does not close, the given system is equivalent to a single resultant represented in magnitude, direction, and sense by the closing line ; its position is obtained from the funicular polygon whose initial and final lines must intersect on the resultant.

If, however, the force polygon closes, the system may be equivalent to a couple, or it may be in equilibrium. The distinction between these two cases is indicated by the funicular polygon. If the initial and final lines of this polygon coincide, the system is in equilibrium ; if they are merely parallel, these lines are the directions of the forces of the couple to which the whole system reduces. The magnitude and sense of the forces of the resulting couple are obtained from the force polygon.

174. Thus it follows from the graphical as well as from the analytical method that *a plane system may be equivalent to a single force, or to a couple, or to zero.* In the first case, the force polygon does not close, and the initial and final sides of the funicular polygon intersect at a finite distance. In the second case, the force polygon closes, and the initial and final lines of the funicular polygon are parallel. In the third case, the force polygon closes, and the initial and final sides of the funicular polygon coincide.

The *graphical conditions of equilibrium* of a plane system are, therefore, two : (1) the force polygon must close ; (2) the funicular polygon must have its initial and final sides coincident.

175. To every *vertex* of the force polygon corresponds a *side* of the funicular polygon, and *vice versa*. The force polygon is said to close if the last vertex coincides with the first ; similarly, the funicular polygon might be said to close when its last side

coincides with the first. With this convention, we may say that *the conditions of equilibrium of a plane system require the closing of both the force polygon and the funicular polygon.*

176. One of the most important applications of the graphical methods is found in the *determination of the stresses in the frame-works* used for bridges, roofs, cranes, etc. The following example will illustrate the method.

Fig. 51 represents the skeleton frame of a roof truss subjected to the "loads" W_1 , W_2 , W_3 and the reactions of the supports A , B . The members of the frame in connection with the lines of action of these forces (imagined as drawn from infinity up to the points of application) divide the whole plane into a number of compartments marked in the figure by the letters a , b , c , d , ... The external forces as well as the members of the frame (or the stresses acting along them) can thus be designated by the two letters of the two portions of the plane separated by the force or stress. For instance, the reaction A is denoted ab , and the stresses in the two members concurring at A are bc and ca . The figure just described may be called the *frame diagram*; and we proceed now to construct its *stress diagram*.*

Laying off on a vertical line $be = W_1$, $eg = W_2$, $gj = W_3$, and bisecting bj at a , we have the polygon of the external forces which gives the reactions $A = ab$, $B = ja$.

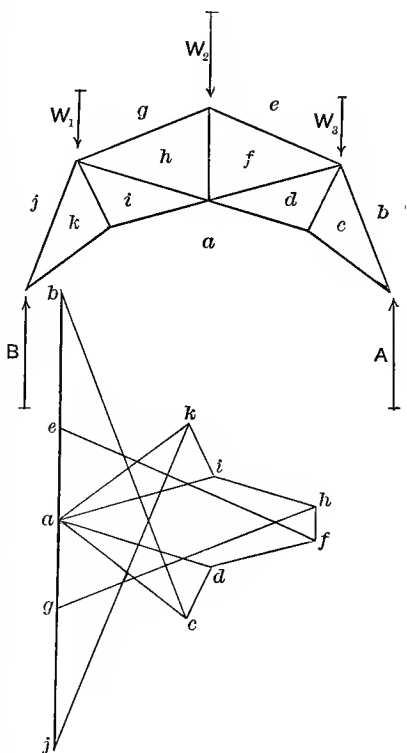


Fig. 51.

* The student is advised to draw the stress diagram himself step by step as indicated in the text.

Next, beginning at the vertex A the stresses in the two members intersecting at A are found by resolving the reaction A along the directions of these members; and this is done in the stress diagram by drawing parallels to these directions through the points a and b . The intersection is denoted by c .

177. It will be noticed that the three lines meeting at A have corresponding to them, in the stress diagram, the three sides ab , bc , ca of a triangle. The force $A = ab$ is represented by ab ; the stress in the member bc (*i.e.* in the member separating the compartment b , c in the frame diagram) is represented in magnitude, direction, and sense by the side bc in the stress diagram; and the stress in the member ca is given by the side ca of the triangle abc . To obtain the sense of each stress correctly, the triangle abc in the stress diagram must be traversed in the sense of the known force $A = ab$; this shows that the member bc is compressed, the stress at A acting towards A , while ca is subject to tension.

It will be found in general that *the lines of the stress diagram corresponding to all the lines meeting at any one vertex of the frame diagram form a closed polygon*. The reason is obvious: the forces at the vertex must be in equilibrium.

178. To continue the construction of the stress diagram, we pass to another vertex of the frame diagram, selecting one at which not more than two stresses are unknown. Thus at the vertex acd the stress in ac is known, being represented by ac in the stress diagram. Hence drawing through a a parallel to da , through c a parallel to cd , we find the point d of the stress diagram.

The vertex $dcbe$ can now be attacked; dc , cb , be are already drawn, and it only remains to draw ef parallel to ef and df parallel to df .

The rest explains itself. Considerations of symmetry are frequently helpful in affording checks.

179. Exercises.

(1) Check the computed stresses of Exercises (1) and (2), Art. 172, by constructing the stress diagrams.

(2) Find the stresses in the frame (Fig. 52), if the load consists of seven equal weights, of 2 tons each, applied at the joints of the upper chord. Owing to the symmetry of the figure, it is sufficient to construct

the stress diagram for half the frame. At the vertex F , a difficulty arises, there being apparently three members whose stresses are not

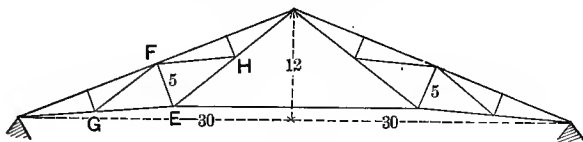


Fig. 52.

known from the previous construction ; but on account of the symmetry with respect to EF , the members FG and FH must have equal stresses.

180. Shearing Force and Bending Moment. Consider a horizontal beam fixed at one end A (Fig. 53), and acted upon at the other end B by a vertical force F . If the beam be cut at any point C of its length, and the equilibrium of the portion AC be considered, the action on AC of the portion removed must be replaced by its equivalent. Now the force F at B is equivalent, by Art. 136, to an equal and parallel force F at C in connection with a couple whose moment is $F \cdot BC$.

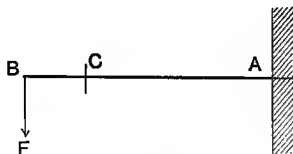


Fig. 53.

The force F at C is called the **shearing force** of the cross-section C , and the moment $F \cdot BC$ the **bending moment** at C . Both are of great importance in engineering, as their combined effect represents what must be overcome by the resistance of the material of the beam, *i.e.* by the internal forces holding together its fibres.

These definitions are readily generalised. Let any beam or girder, supported in any manner, and acted upon by any number of vertical forces, be divided by a vertical cross-section into two portions A and B . For the portion A the shearing force at the cross-section is the sum of all the external forces acting on B ; and the bending moment is the sum of the moments of all these forces with respect to some point in the cross-section.

181. According to its definition the bending moment of a beam at any cross-section is found by adding the moments, with respect to the cross-section, of all the external forces on one side of the section.

Graphically, the bending moment is readily derived from the funicular polygon. Thus in Fig. 54, for the cross-section $\alpha\beta$, the resultant of the forces on the left is $R' = A - W_1 - W_2 = 0.3$ in

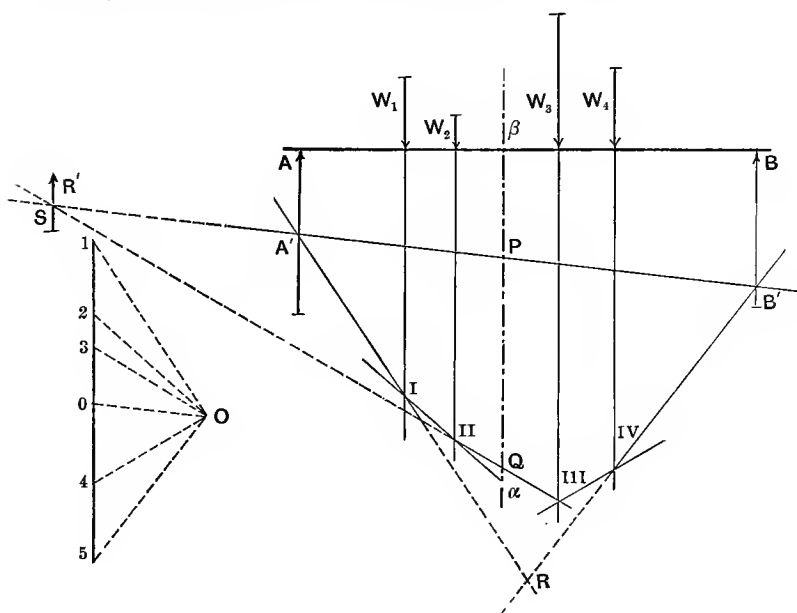


Fig. 54.

the force polygon. Its position is found by bringing to intersection the two sides $A'B'$ and II III of the funicular polygon met by the section $\alpha\beta$. For the funicular polygon resolves A along $A'B'$ and $A'I$, W_1 along IA' and I II, W_2 along II I and II III. The components falling into the same line being equal and opposite (as appears from the force polygon), the forces A , W_1 , W_2 are together equivalent to the components along $A'B'$ and II III; their resultant R' must therefore pass through the intersection S of these lines.

Now if p be the horizontal distance of the point S from $\alpha\beta$, the bending moment at $\alpha\beta$ is $R' \cdot p = o_3 \cdot p$. If $\alpha\beta$ intersect $A'B'$ in P , II III in Q , the triangles SPQ and Oo_3 are similar, so that their altitudes p and H are as the homologous sides PQ and o_3 ; hence

$$p = \frac{PQ}{o_3} \cdot H,$$

and the value of the bending moment is $H \cdot PQ$. As H is constant, we find that *the bending moment is proportional to the vertical height, or ordinate, of the funicular polygon.*

5. FRICTION.

182. The reaction between two surfaces in contact has so far been regarded as directed along the common normal of the surfaces. This is true when the surfaces are *perfectly smooth*.

The surfaces of physical bodies are *rough*, i.e. they present small elevations and depressions; when two such surfaces are "in contact" the projections of one will more or less enter into depressions of the other; the greater the normal pressure between the surfaces, the more will this be the case. Hence when a tangential force acting on one of the bodies tends to *slide* its surface over that of the other body, a resistance will be developed whose magnitude must depend on the roughness of the surfaces and on the normal pressure between them. This resistance is called the **force of friction**.

The study of friction belongs properly to applied mechanics, and will here only be touched upon very briefly.

183. Imagine a body resting with a plane surface on a horizontal plane. Let a small horizontal force P be applied at its centroid (which is supposed to be situated so low that the body is not overturned), and let the force P be gradually increased until motion ensues. The value of P when motion just begins is equal and opposite to the *frictional resistance* F between the

surfaces at this moment, and this resistance is called the **limiting static friction**.

Careful experiments have shown this force to be subject to the following laws :

(1) *The magnitude of the limiting friction F bears a constant ratio to the normal pressure N between the surfaces in contact; that is*

$$F = \mu N,$$

where μ is a constant depending on the condition and nature of the surfaces in contact. This constant which must be determined experimentally for different substances and surface conditions is called the **coefficient of static friction**. It is in general a proper fraction ; for perfectly smooth surfaces $\mu = 0$.

(2) *For a given normal pressure the limiting static friction, and hence the coefficient of static friction, is independent of the area of contact.*

184. The frictional resistance between two surfaces in relative motion is called **kinetic friction**. It is subject, in addition to the two laws just mentioned, to the third law :

(3) *Kinetic friction is independent of the velocities of the bodies in contact.*

The coefficient of static friction is generally slightly greater than that of kinetic friction.

It must not be forgotten that these so-called **laws of friction** are experimental laws, and therefore true only approximately, and within the limits of the experiments from which they were deduced. When the relative velocity of the surfaces in contact is very high, and when, as is usually the case in machinery, a lubricating material is introduced between the two surfaces, the frictional resistance is found to depend on a number of other circumstances, such as the temperature, the form of the surfaces, the velocity, the nature of the lubricator, etc.

185. Consider again a body resting on a horizontal plane (Fig. 55), and acted upon by a horizontal force P just large

enough to equal the limiting friction F . The normal reaction N of the plane is equal and opposite to the weight W . The body is thus in equilibrium under the action of the two pairs of equal and opposite forces; but motion will ensue as soon as P is increased. If P be decreased, F will decrease at the same rate, so that the equilibrium remains undisturbed.

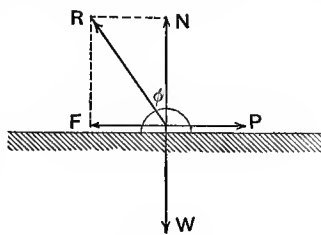


Fig. 55.

The force of friction F can be combined with the normal reaction N to form a resultant,

$$R = \sqrt{F^2 + N^2} = \sqrt{P^2 + W^2},$$

which represents the *total reaction* of the horizontal plane.

If ϕ be the angle between N and R when F has its limiting value $F = \mu N$ (Art. 183), we have, since $\tan \phi = F/N$,

$$\tan \phi = \mu.$$

The angle ϕ thus gives a kind of graphical representation for the coefficient of friction μ ; it is called the **angle of friction**.

186. If the plane be not horizontal, but inclined to the horizon

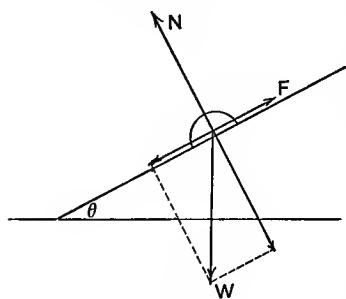


Fig. 56.

at an angle θ , the weight W of the body (regarded as a particle) resting on the plane can be resolved into a component $W \sin \theta$ along the plane, and a component $W \cos \theta$ perpendicular to it (Fig. 56). Hence, if no other forces act on the body it will be in equilibrium, provided the component $W \sin \theta$ be not greater than the

limiting friction $F = \mu W \cos \theta$. The limiting condition of equilibrium is, therefore,

$$\mu W \cos \theta = W \sin \theta, \quad \text{or} \quad \mu = \tan \theta;$$

in other words, if the angle θ be gradually increased, the body will not slide down the plane until $\theta > \phi$. This furnishes an experimental method of determining the angle of friction ϕ , which on this account is sometimes called the **angle of repose**.

187. A particle P (Fig. 57) will be in equilibrium on any rough surface, if the total reaction of the surface, *i.e.* the resultant R of the normal reaction N and the friction F , is equal and opposite to the resultant R' of all the other forces acting on the particle.

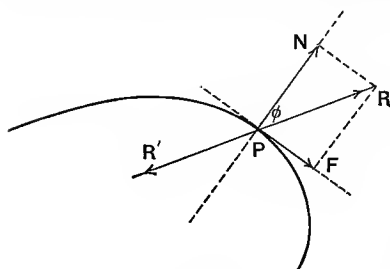


Fig. 57.

The limiting value of the angle between N and R is ϕ , so that the particle can be in equilibrium only if the resultant R' makes with the normal an angle $\leq \phi$. Hence, if about the normal PN as axis, and with P as vertex, a cone be described whose vertical angle is 2ϕ , the condition of equilibrium is that R' must lie within this cone.

The cone is called the **cone of friction**.

188. Exercises.

(1) A particle of weight W is in equilibrium on a rough plane inclined to the horizon at an angle θ , under the action of a force P parallel to the plane along its greatest slope. Determine P : (a) when $\theta > \phi$, (b) when $\theta = \phi$, (c) when $\theta < \phi$, $\phi = \tan^{-1} \mu$ being the angle of friction.

(2) Determine the tractive force required to haul a train of 100 tons with constant velocity up a grade of 2.5 per cent if the coefficient of friction is $1/200$.

(3) A weight W is to be hauled along a horizontal plane, the coefficient of friction being $\mu = \tan \phi$. Determine the required tractive force P if it is to act at an inclination α to the horizon, and show that this force is least when $\alpha = \phi$.

(4) A particle of weight W is kept in equilibrium on a plane inclined at an angle θ to the horizon by a force P making an angle α with the line of greatest slope (in the vertical plane at right angles to the intersection of the inclined plane with the horizon). Find the conditions of equilibrium when the particle is on the point of moving (a) down the plane, (b) up the plane.

(5) A homogeneous straight rod $AB = 2l$ of weight W rests with one end A on the horizontal floor, with the other end B against a vertical wall whose plane is at right angles to the vertical plane of the rod. If there be friction of angle ϕ at both ends, determine the limiting position of equilibrium.

(6) Two particles whose weights are W, W' are in equilibrium on an inclined plane, being connected by a string directed along the line of greatest slope. If the coefficients of friction are μ, μ' , determine the inclination of the plane.

189. The idea of the angle of friction suggests a *graphical method* for problems on equilibrium with friction.

The example of a rod resting on two inclined planes, Art. 150, Fig. 41, may serve as an example. If the intersection E of the normal reactions A and B lies on the vertical through D , the rod will be in equilibrium whether there be friction at A and B or not. When this condition is not fulfilled, the rod may still be in equilibrium if there be sufficient friction between the ends of the rod and the supporting planes.

Let $\mu = \tan \phi$ be the coefficient of friction on the plane CA , $\mu' = \tan \phi'$ that on CB ; then the total reactions at A and B will, by Art. 185, make angles not greater than ϕ and ϕ' , respectively, with the normals to the planes. Hence the two limiting positions of equilibrium for the weight W , in a given position of the rod, can be found by bringing the lines of these total reactions to intersection; the limiting position of W is the vertical through this intersection. Thus, to prevent the rod from sliding up the plane CA and down the plane CB , the friction angles ϕ, ϕ' must be applied in the negative sense (clockwise) to the normals at A and B ; this gives one limiting position D' for the point D . The other position D'' is found by applying the friction angles in the positive sense. Equilibrium will therefore subsist if the weight be placed anywhere between D' and D'' .

The construction is somewhat simplified when $\phi = \phi'$ since then the intersections of the total reactions lie on the circle described about ABC (Fig. 58).

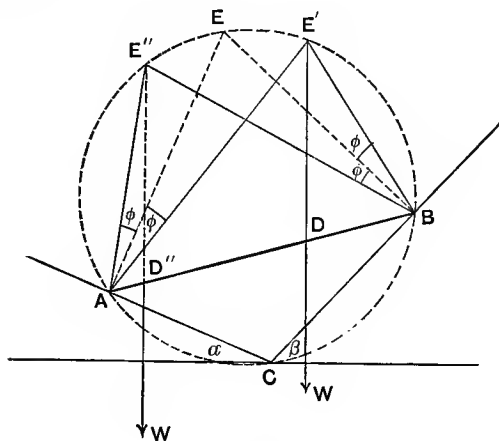


Fig. 58.

190. As another example consider the ordinary jack intended to raise an eccentric load W acting vertically downwards through A (Fig. 59) by a force P passing vertically upwards through the pitch-line B of the rack. Near C and D the rack is pressed against the casing. The directions of the total reactions C, D at these points are found by applying the friction angle to the normals.

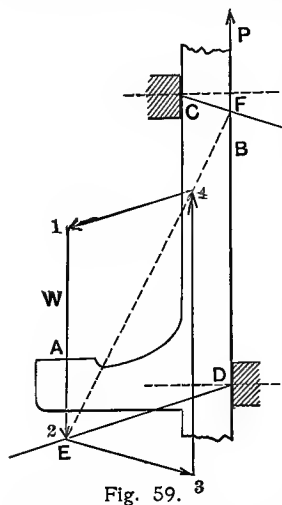


Fig. 59.

The four forces W, P, C, D can be in equilibrium only if the resultant of W and D is equal and opposite to the resultant of P and C ; hence, if E be the intersection of W and D , F that of P and C , each of these resultants must act along EF .

If the load W be known, the other forces can now be found by constructing the force polygon. Draw $12 = W$ in position (*i.e.* through A); draw 23 parallel to C ; 41 parallel to D ; and through the intersection 4 of 41 with EF draw the vertical 34 to the intersection 3 with 23 .

191. Journal Friction. A journal, or trunnion, is the cylindrical end of a horizontal shaft, by means of which the shaft is supported in its bearing. The larger circle in Fig. 60 represents a cross-section of the journal at right angles to the axis of the shaft.

The shaft and journal may be regarded as rotating uniformly about their common horizontal axis under the action of a driving force whose moment with respect to a point O on the axis would have to be exactly equal and opposite to that of the resistance, or load, if there were no journal friction. For, in this case, the reaction of the bearing to the weight W of the shaft would act vertically upwards through the axis of the shaft, so that its moment would be zero.

The existence of friction at the place of contact A between journal and bearing requires an increase of the driving force, which may be regarded as a small tangential force P applied at any point B , such that its moment about O of the frictional resistance at A .

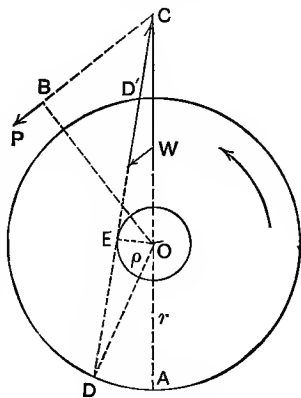


Fig. 60.

192. Let C be the intersection of the direction of this force P with the vertical through O and A , which is the line of action of the weight W of the shaft. The resultant of P and W passes through C , and intersects the circumference of the journal in a point D near A ; the total reaction of the bearing is equal and opposite to this resultant. As the total reaction must make an angle equal to the angle of friction ϕ with the normal at D which passes through the centre O , we have for the perpendicular OE dropped from O on CD ,

$$OE = \rho = r \sin \phi,$$

where r is the radius of the journal. A circle described about O , with ρ as radius, has the total reaction of the bearing as a tangent. This circle is called the **friction circle**. As ϕ is generally very small in the case of journal friction, $\mu = \tan \phi$ can be substituted for $\sin \phi$, and we have for the radius ρ of the friction circle

$$\rho = \mu r.$$

As soon as any one point is known through which the total reaction

must pass (as the point C in Fig. 60), its direction is found by drawing through this point a tangent to the friction circle.

193. If the shaft revolved in the opposite sense, *i.e.* clockwise (instead of counter-clockwise, as assumed in Fig. 60), the tangent to the friction circle would have to be drawn through C on the *other* side of the friction circle.

In the case of **axle-friction**, *i.e.* when the journal, or axle, is fixed, while the bearing, or hub, revolves about it, the same considerations would apply, except that the point of application of the total reaction would now be at the top, at D' , instead of D .

194. Pin-friction, as it occurs in link-work and jointed frames that are not absolutely stiff, is not different from journal friction or axle-friction, and can be treated in the same way. Thus, a link connected to other parts of a machine by means of a pin at each end would transmit the force along the line joining the centres of the pins if there were no friction. To take account of pin-friction, we have only to draw the friction circles about the centre of each pin; the direction in which the force is transmitted by the link is tangent to both these circles.

Which one of the four common tangents represents this direction must be decided in each particular case by considering that the reaction exerted by one link on another connected with it by a pin is in the direction of the motion of the former relative to the other. Thus if the link AB (Fig. 61) be subject to tension, and its motion relative to

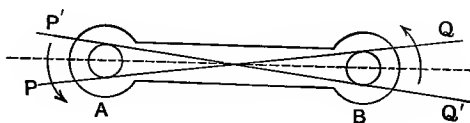


Fig. 61.

the adjoining links at A and B be as indicated by the arrows in the figure, the contact between the link and pin will be on the outside both at A and at B ; the friction is, therefore, directed downwards at A and upwards at B , and the line PQ along which the force is transmitted touches the friction circle at A below, at B above.

If the link were under compression, with the same relative motions, the line of force would have the direction $P'Q'$.

195. The simplest case of **pivot friction** is that of a vertical shaft of weight W resting with its circular end on a plane horizontal support. If a be the radius of the end of the shaft, the pressure per unit of area is $W/\pi a^2$, and the pressure on a polar element of area is $\frac{W}{\pi a^2} \cdot r dr d\theta$.

The friction at this element, $\mu \frac{W}{\pi a^2} \cdot r dr d\theta$, is directed along the tangent to the circle of radius r ; its moment with respect to the centre O of the circle is therefore $\frac{\mu W}{\pi a^2} r^2 dr d\theta$. Hence the whole moment of friction about O is

$$\frac{\mu W}{\pi a^2} \int_0^{2\pi} d\theta \int_0^a r^2 dr = \frac{2}{3} \mu W a = \mu W \cdot \frac{2}{3} a.$$

This may be regarded as the moment of a force μW applied at a distance $\frac{2}{3} a$ from the centre.

196. Belt-friction. A belt running over two pulleys and stretched so tight as to prevent slipping is a common means of transferring the rotary motion about the axis of one pulley, say A , to the axis of the other pulley B ; A is called the driver, B is the driven pulley. We assume the axes parallel and the rotation counter-clockwise.

When the pulleys are at rest the tension in CE (Fig. 62) is of course

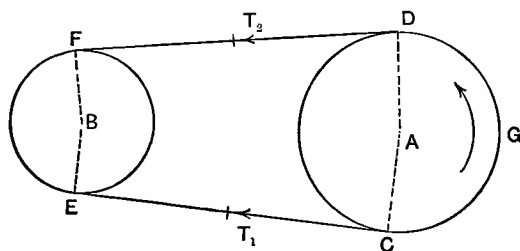


Fig. 62.

equal to the tension in DF . But if the pulley A be set in motion, say by a tangential driving force P acting at a lever-arm p , while the pulley B experiences a resistance Q whose arm is q , the tension in CE will increase to a certain value T_1 , and the tension in DF will decrease to a value T_2 until the difference $T_1 - T_2$ is sufficient to overcome the resistance Q . This difference is due to this friction along the surface CGD . If the resistance Q be too great this friction might not be sufficient, and slipping of the belt on the driver would occur.

197. Let us try to determine the condition which T_1 and T_2 must satisfy to prevent slipping. To do this we determine the equilibrium of the belt at the moment when slipping is just on the point of taking place.

The tension of the belt decreases gradually along the arc CGD from the value T_1 at C to the value T_2 at D . Let it be T at the point P and $T + dT$ at the near point P' (Fig. 63). The portion PP' of the belt is in equilibrium under the action of the forces T , $T + dT$ and the reaction dR of the pulley; hence dR must pass through the intersection of T and $T + dT$ and must make with the radius AP an angle equal to the friction angle ϕ . Resolving these forces along T and at right angles to it, we have, if $\angle PAP' = d\theta$,

$$T + dR \sin \phi = T + dT,$$

$$dR \cos \phi = (T + dT)d\theta,$$

$$\text{or } dR \sin \phi = dT,$$

$$dR \cos \phi = T d\theta;$$

hence, dividing,

$$\tan \phi = \frac{1}{T} \frac{dT}{d\theta}.$$

Putting μ for $\tan \phi$ and integrating over the whole arc of contact, we find, if θ be the angle of this arc,

$$\log T_1 - \log T_2 = \mu \theta,$$

or

$$\frac{T_1}{T_2} = e^{\mu \theta}.$$

For the common system of logarithms this becomes

$$\log \frac{T_1}{T_2} = 0.4343 \mu \theta,$$

where θ must be expressed in circular measure.

198. Rolling Friction. The resistance offered by a surface to the *rolling* of another surface over it is of a somewhat different nature from that of ordinary or *sliding* friction. In sliding friction, the same point or surface area of one body comes in

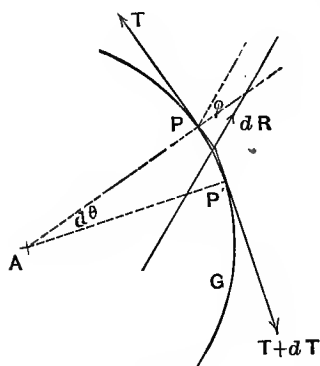


Fig. 63.

contact with different points or areas of the other. In the case of rolling friction, the points that come successively in contact are different for *both* bodies.

Let us examine the simplest case, viz. that of a cylinder rolling over a horizontal plane. If both cylinder and plane were perfectly rigid, there could be no resistance to rolling. This

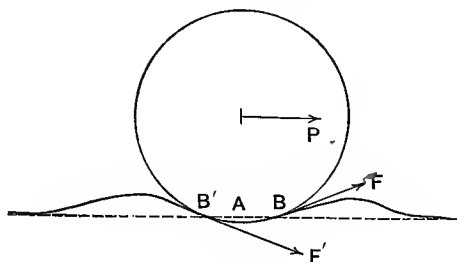


Fig. 64.

resistance is due to the compression both of the lower part of the cylinder and of the plane. Experiments made with a heavy roller on india-rubber have shown that the supporting surface when elastic is not only compressed under the roller but bulges out in front and behind, as indicated in Fig. 64. Thus, the area of contact is considerably increased, and as the roller advances, the portion AB of its surface rubs over the surface of the support, while the elastic material of the support in trying to regain its horizontal surface causes friction over the area $B'A$ also.

The experiments indicate for the value F of rolling friction an expression of the form

$$F = \mu' \frac{W}{r},$$

where W is the weight, r the radius of the cylinder, and μ' a constant depending on the nature of the materials in contact. For hard surfaces, this constant of rolling friction μ' is very much smaller than the constant of sliding friction μ .

199. On the subject of plane statics the student may consult in particular the recent work : E. J. ROUTH, *A treatise on analytical statics*,

with numerous examples, Vol. I., Cambridge, University Press, 1891 ; also G. M. MINCHIN, *A treatise on statics with applications to physics*, Vol. I., 3d ed., Oxford, Clarendon Press, 1884 ; I. TODHUNTER, *Analytical statics*, 5th ed. by J. D. Everett, London, Macmillan, 1887 ; B. PRICE, *Infinitesimal calculus*, Vol. III. : Statics and dynamics of material particles, 2d ed., Oxford, Clarendon Press, 1868. For problems, see also W. WALTON, *Collection of problems in illustration of the principles of elementary mechanics*, 2d ed., Cambridge, Deighton, 1880.

Numerous applications to civil and mechanical engineering will be found in J. H. COTTERILL, *Applied mechanics*, London, Macmillan, 1884 ; W. J. M. RANKINE, *A manual of applied mechanics*, 9th ed. by E. F. Bamber, London, Griffin, 1877 ; A. RITTER, *Lehrbuch der technischen Mechanik*, 5th ed., Leipzig, Baumgärtner, 1884 ; J. WEISBACH, *Mechanics of engineering*, Vol. I., translated by E. B. COXE, New York, Van Nostrand, 1875 ; and in works on graphical statics.

On friction, see in particular : G. HERRMANN, *The graphical statics of mechanism*, translated by A. P. Smith, 2d ed., New York, Van Nostrand, 1892 ; R. H. THURSTON, *Treatise on friction and lost work in machinery and mill work*, New York, Wiley, 1885 ; J. H. JELLETT, *The theory of friction*, Dublin, Hodges, 1872.

VI. *Solid Statics.*

I. THE CONDITIONS OF EQUILIBRIUM.

200. The equilibrium of a rigid body in the most general case, that is, when acted upon by any number of forces F in a space of three dimensions, can be investigated in a manner similar to that adopted for the plane system in Art. 139.

Selecting as origin any point O rigidly connected with the body, let two equal and opposite forces $F, -F$ be applied at O , for every one of the given forces F (Fig. 65). The effect of the given system of forces on the body is not changed by the introduction of these forces at O . But we may now regard the given force F acting at its point of application P as replaced by the equal and parallel force F at O , in combination with

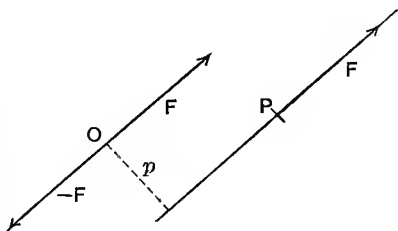


Fig. 65.

the couple formed by the original force F at P and the force $-F$ at O . All the forces of the given system are thus transferred to a common point of application O , and can therefore be compounded into a single resultant R , passing through O and represented in magnitude and direction by the geometric sum of the forces. In addition to this resultant R , we obtain as many couples $(F, -F)$ as there were forces given; and their resultant is found by geometrically adding the vectors of the couples (Art. 134).

Thus the given system of forces is seen to be equivalent to a resultant R in combination with a couple whose vector we shall call H ; in other words, it has been proved that *any system of forces acting on a rigid body can be reduced to a single resultant force in combination with a single resultant couple.*

201. A further reduction is in general not possible. The general *conditions of equilibrium* are, therefore,

$$R=0, \quad H=0.$$

202. Under special conditions it may of course happen that R is perpendicular to the vector H . In this case R and H combine to a single force R (Art. 135), and if the origin be taken on the line of this force, the whole system reduces to a single resultant.

203. It is to be noticed that in the general reduction of forces (Art. 200), the magnitude, direction, and sense of the resultant force R are entirely independent of the position of the origin O , the resultant being simply the geometric sum of all the given forces. The resultant couple H , on the other hand, will in general differ according to the origin selected.

To investigate this dependence, let R, H (Fig. 66) be the *elements of reduction* for the origin O ; i.e. let R be the resultant,

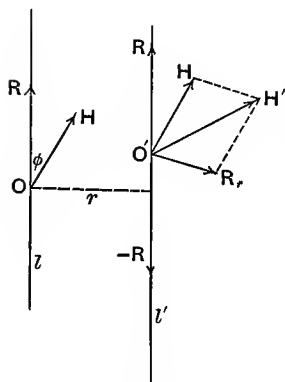


Fig. 66.

H the vector of the resulting couple of a given system of forces when O is selected as origin. To find the elements of reduction of the same system of forces when some other point O' is taken as origin, it is only necessary to apply at O' two equal and opposite forces $R, -R$, each equal and parallel to the original resultant R . The given system of forces being equivalent to R and H at O will also be equivalent to the resultant R at O' , the couple whose vector is H (which may be

drawn through O' without changing its effect), and the couple formed by R at O and $-R$ at O' . If l be the line of R through O , l' the line of R through O' , and r the distance of these parallels, the moment of the latter couple is Rr and its vector is at right angles to the plane (l, O'). Combining the vectors H

and Rr into a resultant vector H' by geometric addition, we have found the elements of reduction R, H' for the origin O' .

204. If the new origin O' had been selected on the line l of the original resultant, no new couple (R, r) would have been introduced, and H would not have been changed. But whenever the line of action l of the resultant is changed, the vector of the resultant couple H is changed.

By increasing the distance r between l and l' the moment Rr of the additional couple is increased. The effect of combining this additional couple Rr with H is, in general, to vary both the magnitude of the resulting couple H' and the angle ϕ it makes with the direction of the resultant R . It can be shown that the line l' of the new resultant can always be selected so as to reduce the angle ϕ to zero. The line l_0 for which $\phi=0$, *i.e.* for which the vector H of the resultant couple is parallel to the resultant force R , is called the **central axis** of the given system of forces. We proceed to show how it can be found.

205. Let the vector H be resolved at O into a component $H_0=H \cos \phi$ along l , and a component $H_1=H \sin \phi$, at right angles to l (Fig. 67). In the plane passing through l at right angles to H_1 , it is always possible to find a line l_0 parallel to l at a distance r_0 from l , such as to make $Rr_0 = -H_1$.

The line l_0 so determined is the central axis. For, if this line be taken as the line of the resultant R , the additional couple Rr_0 destroys the component H_1 , so that the resulting couple H_0 has its vector parallel to R .

206. As the direction of the vector H is always changed in passing from line to line, there can be but one central axis for a given system of forces.

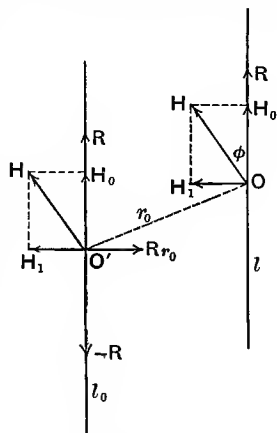


Fig. 67.

It appears from the construction of the central axis given in Art. 205, that the vector of the resulting couple for this axis l_0 is $H_0 = H \cos \phi$; it is, therefore, less than for any other line.

It is instructive to observe how the vector H increases and changes its direction as we pass from the central axis l_0 to any parallel line l .

The transformation from l_0 to l requires the introduction of a couple (R, r_0) whose vector Rr_0 (Fig. 68) is at right angles to the plane (l_0, l) and combines with H_0 to form the resulting

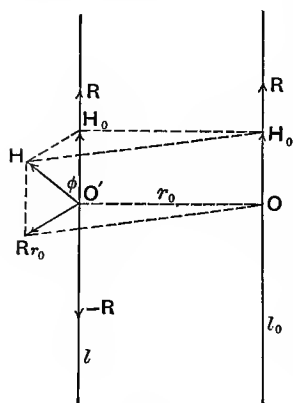


Fig. 68.

couple H for l . As the distance r_0 of l from l_0 is increased, both the magnitude of H and the angle ϕ it makes with l increase until, for an infinite r_0 , the angle ϕ becomes a right angle.

207. It is evident that since $H_0 = H \cos \phi$, the product $RH \cos \phi$ is a constant quantity for a given system of forces. It has been called the **invariant** of the system.

If the elements of reduction for the central axis (R, H_0) be given, those for any parallel line l at the distance r_0 from the central axis are determined by the equations

$$H^2 = H_0^2 + R^2 r_0^2, \quad \tan \phi = \frac{R r_0}{H_0}.$$

208. To sum up the results of the preceding articles, it has been shown that *any system of forces acting on a rigid body can be reduced, in an infinite number of ways, to a resultant R in combination with a couple H* . For all these reductions the magnitude, direction, and sense of the resultant R are the same, but the vector H of the couple changes according to the *position* assumed for the line of R . There is one, and only one, position of R , called the central axis of the system, for which the vector

H is parallel to R , and has at the same time its least value, H_0 ; this value H_0 is equal to the projection of any other vector H on the direction of the resultant R .

209. While, in general, a system of forces cannot be reduced to a single resultant, it can always be reduced to *two non-intersecting forces*. This easily follows by considering the system reduced to its resultant R and resulting couple H for any origin O (Fig. 69). Let F , $-F$ be the forces, p the arm of the couple H , and place this couple so that one of the forces, say $-F$, intersects R at O . Then, combining R and $-F$ to their resultant F' , the given system of forces is evidently equivalent to the two non-intersecting forces F , F' (compare Art. 137).

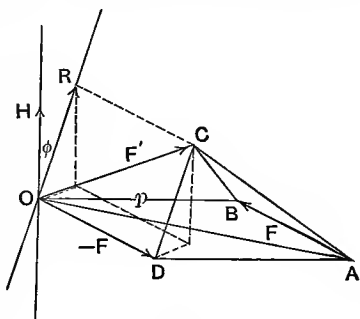


Fig. 69.

210. The two forces F , F' determine a tetrahedron $OABC$; and it can be shown that *the volume of this tetrahedron is constant and equal to one sixth of the invariant of the system* (Art. 207). The proof readily appears from Fig. 69. The volume of the tetrahedron $OABC$ is evidently one half of the volume of the quadrangular pyramid whose vertex is C and whose base is the parallelogram $OBAD$. The area of this parallelogram is $Fp = H$; and the altitude of the pyramid is $= R \cos \phi$, being equal to the perpendicular let fall from the extremity of R on the plane of the couple; hence the volume of the tetrahedron

$$= \frac{1}{6} RH \cos \phi = \frac{1}{6} RH_0.$$

211. To effect the reduction of a given system of forces analytically, it is usually best to refer the forces F and their points of application P to a rectangular system of co-ordinates

Ox, Oy, Oz (Fig. 70). Let x, y, z be the co-ordinates of P and X, Y, Z the components of F parallel to the axes.

To transfer these components to O as common origin, we proceed similarly as in Art. 142. Thus to transfer, say X , we introduce at P' , the foot of the perpendicular let fall from P on the plane zx , two equal and opposite forces $X, -X$; and we do the same thing at O . Then the single force X at P is replaced by the force X at O in combination with the two couples formed by X at $P, -X$ at P' , and X at $P', -X$ at O . The vector of the former couple is parallel to Oz , its moment is $-yX$; the negative sign being used because for a person looking on the plane of the couple from the positive side of the axis Oz the

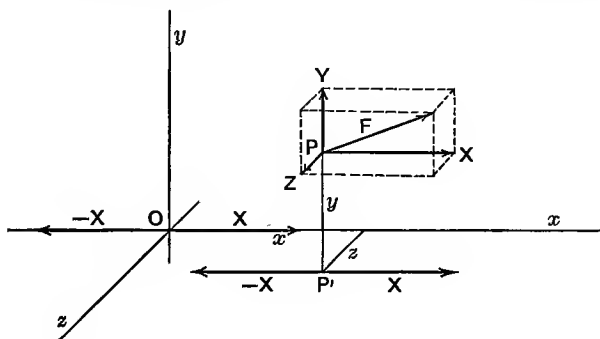


Fig. 70.

couple rotates clockwise. The vector of the latter couple is parallel to Oy , and its moment is zX .

The transfer of Y to the origin O requires the introduction of two couples, $-zY$ having its vector parallel to Ox , and xY having its vector parallel to Oz .

Finally, transferring Z to O , we have to introduce the couples $-xZ$ with a vector parallel to Oy , and yZ with a vector parallel to Ox .

Thus each force F is replaced by three forces X, Y, Z along the axes of co-ordinates and applied at O , in combination with three couples whose vectors are $yZ - zY$ parallel to Ox , $zX - xZ$ parallel to Oy , $xY - yX$ parallel to Oz .

212. Doing the same thing for every force of the given system and adding the components having the same direction, the system will be found equivalent to the three rectangular forces

$$\Sigma X, \quad \Sigma Y, \quad \Sigma Z$$

applied at O , together with the three couples

$$\Sigma (yZ - zY), \quad \Sigma (zX - xZ), \quad \Sigma (xY - yX),$$

whose vectors are at right angles.

The three forces can now be compounded into a single resultant

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2},$$

whose direction is determined by the angles α, β, γ which it makes with the axes Ox, Oy, Oz ,

$$\cos \alpha = \frac{\Sigma X}{R}, \quad \cos \beta = \frac{\Sigma Y}{R}, \quad \cos \gamma = \frac{\Sigma Z}{R}.$$

In the same way the three couples can be compounded into a single resulting couple whose moment is

$$H = \sqrt{[\Sigma (yZ - zY)]^2 + [\Sigma (zX - xZ)]^2 + [\Sigma (xY - yX)]^2}.$$

213. Since R^2 , as well as H^2 , is thus found as the sum of three squares, each of these quantities can vanish only if the three squares composing it vanish separately. The **conditions of equilibrium of a rigid body** (Art. 201) are therefore expressed analytically by the following six equations:

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

$$\Sigma (yZ - zY) = 0, \quad \Sigma (zX - xZ) = 0, \quad \Sigma (xY - yX) = 0.$$

As the system of co-ordinates can be selected arbitrarily, the meaning of the first three equations is that the sum of the components of all the forces along any three lines not in the same plane must vanish. The last three equations express that the

sum of the moments of all the forces about any three axes not in the same plane must also vanish. The *moment of a force about an axis* must be understood as meaning the moment of its projection on a plane at right angles to the axis with respect to the point of intersection of the axis with the plane. This definition is in accordance with the somewhat vague notion of the moment of a force as representing its "turning effect." For, regarding the force as acting on a rigid body with a fixed axis, the force can be resolved into two components, one parallel, the other perpendicular, to the axis; the former component does evidently not contribute to the turning effect, which is therefore measured by the moment of the latter alone.

214. The *equations of the central axis* (Art. 204) can be found by a transformation of co-ordinates.

Let the system be reduced for any origin O to its resultant R , whose rectangular components we denote by

$$A = \Sigma X, \quad B = \Sigma Y, \quad C = \Sigma Z,$$

and to the vector H of its resulting couple with the components

$$L = \Sigma (yZ - zY), \quad M = \Sigma (zX - xZ), \quad N = \Sigma (xY - yX).$$

If a point O' whose co-ordinates are ξ, η, ζ be taken as new origin and the co-ordinates of any point with respect to parallel axes through O' be denoted by x', y', z' , we have $x = \xi + x'$, $y = \eta + y'$, $z = \zeta + z'$. Substituting these values, we find

$$\begin{aligned} L &= \Sigma [(\eta + y')Z - (\zeta + z')Y] = \eta \Sigma Z - \zeta \Sigma Y + \Sigma (y'Z - z'Y) \\ &= \eta C - \zeta B + L', \end{aligned}$$

where L' is the x -component of the couple H' resulting for O' as origin. Similar expressions hold for M and N . The components of H' are therefore

$$L' = L - \eta C + \zeta B, \quad M' = M - \zeta A + \xi C, \quad N' = N - \xi B + \eta A;$$

and its direction cosines are

$$\lambda = \frac{L'}{H'}, \quad \mu = \frac{M'}{H'}, \quad \nu = \frac{N'}{H'}.$$

The central axis being defined (Art. 204), as that line for which the vector of the resulting couple is parallel to the direction of the resultant, the point $O'(\xi, \eta, \zeta)$ will lie on the central axis if the direction cosines of H' are proportional to those of R , viz., to

$$\alpha = \frac{A}{R}, \quad \beta = \frac{B}{R}, \quad \gamma = \frac{C}{R}.$$

Hence the equations of the central axis are

$$\frac{L'}{A} = \frac{M'}{B} = \frac{N'}{C},$$

or

$$\frac{L - \eta C + \zeta B}{A} = \frac{M - \zeta A + \xi C}{B} = \frac{N - \xi B + \eta A}{C}.$$

215. To show the application of the conditions of equilibrium, let us consider the simple machine called the *wheel and axle*. It consists of a horizontal shaft (Fig. 71) resting with its ends on the supports or bearings A, B , and is intended to raise a weight W , suspended vertically by means of a rope wound around the shaft. The driving force F is applied

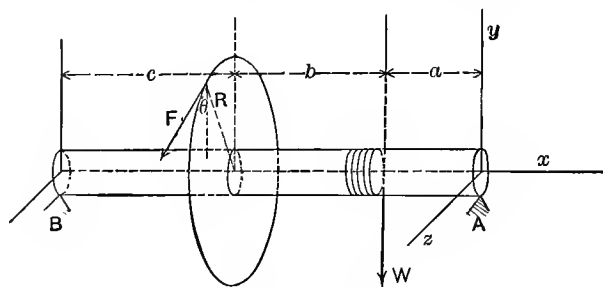


Fig. 71.

on the circumference of the "wheel," i.e. in a vertical plane at right angles to the axis of the shaft. It is required to find the relation between F and W for equilibrium, and the pressures on the bearings A, B .

Let r be the radius of the shaft, R that of the wheel, *i.e.* the lever-arm of the force F , and let F be inclined to the vertical at an angle θ ; then, with the co-ordinates and notations of the figure, the conditions $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$, give

$$A_x + B_x = 0, \quad A_y + B_y - W - F \cos \theta = 0, \quad A_z + B_z + F \sin \theta = 0,$$

where A_x, A_y, A_z are the components of the unknown reaction at A ; B_x, B_y, B_z , those at B .

Taking moments about each of the co-ordinate axes, we find

$$FR = Wr, \quad (a+b)F \sin \theta + lB_z = 0, \quad aW + (a+b)F \cos \theta - lB_y = 0,$$

where $l = a + b + c$ is the length of the shaft.

A_x and B_x must evidently be separately zero. Solving the equations, we find

$$F = \frac{r}{R} \cdot W,$$

$$A_y = \frac{1}{l} \left(b + c + \frac{cr}{R} \cos \theta \right) \cdot W, \quad B_y = \frac{1}{l} \left(a + \frac{(a+b)r}{R} \cos \theta \right) \cdot W,$$

$$A_z = -\frac{1}{l} \frac{cr}{R} \sin \theta \cdot W, \quad B_z = -\frac{1}{l} \frac{(a+b)r}{R} \sin \theta \cdot W.$$

216. As another example, consider a rigid body of weight W , supported at three points A_1, A_2, A_3 ; and let it be required to determine the distribution of the pressure between the three supports.

Let the vertical through the centroid of the body meet the plane of the triangle $A_1A_2A_3$ in a point G , whose distances from the sides A_2A_3, A_3A_1, A_1A_2 we may denote by p_1, p_2, p_3 . Then, if A_1, A_2, A_3 be the unknown reactions, and h_1, h_2, h_3 the altitudes of the triangle, we have

$$A_1 + A_2 + A_3 = W,$$

and, taking moments about A_2A_3, A_3A_1, A_1A_2 ,

$$A_1 \cdot h_1 = W \cdot p_1, \quad A_2 \cdot h_2 = W \cdot p_2, \quad A_3 \cdot h_3 = W \cdot p_3.$$

$$\text{Hence, } A_1 = \frac{p_1}{h_1} W, \quad A_2 = \frac{p_2}{h_2} W, \quad A_3 = \frac{p_3}{h_3} W.$$

Substituting these values into the first equation, we find the condition,

$$\frac{p_1}{h_1} + \frac{p_2}{h_2} + \frac{p_3}{h_3} = 1.$$

If G falls outside the triangle, one or two of the points A_1, A_2, A_3 will be subject to pressures vertically upwards. If G be the centroid of the triangular area $A_1A_2A_3$, we have $p_1/h_1 = p_2/h_2 = p_3/h_3 = 1/3$; hence in this case the three reactions are equal.

217. *The axis of the hinges of a door is inclined at an angle θ to the horizon. The door is turned out of its position of equilibrium by an angle ϕ , and held in this position by a force F perpendicular to the plane of the door. Determine F and the reaction of the hinges A, B (Fig. 72).*

Let the axis of the hinges be taken as the axis of x , the vertical plane through it as the plane zx , and the point midway between the hinges A, B as the origin O . Regarding the door as a homogeneous rectangular plate whose dimensions are $AB = 2a$, $OC = 2b$, the co-ordinates of its centroid G are $0, b \sin \phi, b \cos \phi$. If the force F be applied at a point P on the middle line OC at the distance $OP = p$ from O , the co-ordinates of its point of application P are $0, p \sin \phi, p \cos \phi$.

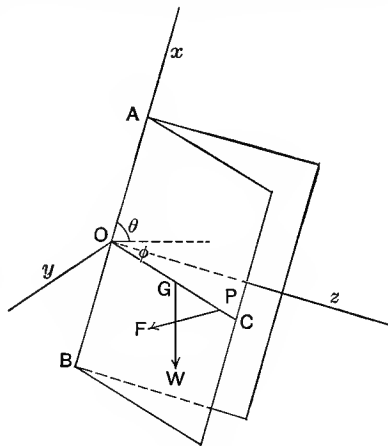


Fig. 72.

To proceed systematically, we may tabulate the components of the forces, and the co-ordinates of their points of application, and then form the component couples, as shown below. The components of the unknown reactions A, B of the hinges are called $A_x, A_y, A_z, B_x, B_y, B_z$.

FORCES.	COMPONENTS.			CO-ORDINATES.			COUPLES.		
	X	Y	Z	x	y	z	$yZ - zY$	$zX - xZ$	$xY - yX$
W	$-W \sin \theta$	0	$W \cos \theta$	0	$b \sin \phi$	$b \cos \phi$	$Wb \cos \theta \sin \phi$	$-Wb \sin \theta \cos \phi$	$Wb \sin \theta \sin \phi$
F	0	$F \cos \phi$	$-F \sin \phi$	0	$p \sin \phi$	$p \cos \phi$	$-Fp(\sin^2 \phi + \cos^2 \phi)$	0	0
A	A_x	A_y	A_z	a	0	0	0	$-A_x a$	$A_y a$
B	B_x	B_y	B_z	$-a$	0	0	0	$B_x a$	$-B_y a$

From this table the six conditions of equilibrium are at once obtained :

$$-W \sin \theta + A_x + B_x = 0, \quad (1)$$

$$F \cos \phi + A_y + B_y = 0, \quad (2)$$

$$W \cos \theta - F \sin \phi + A_z + B_z = 0, \quad (3)$$

$$Wb \cos \theta \sin \phi - Fp = 0, \quad (4)$$

$$-Wb \sin \theta \cos \phi + (-A_x + B_x)a = 0, \quad (5)$$

$$Wb \sin \theta \sin \phi + (A_y - B_y)a = 0. \quad (6)$$

If the reactions were not required, equation (4) alone would be sufficient, as it furnishes the value of F , viz.,

$$F = \frac{b}{p} \cos \theta \sin \phi \cdot W.$$

This relation can of course be found directly by taking moments about the axis of the hinges. It shows that, for a given inclination of the hinges, F is greatest when $\phi = \pi/2$.

The remaining five equations are sufficient to determine $A_x + B_x$, A_y , A_z , B_y , B_z .

To find the reactions for a door with vertical axis, we have to put $\theta = \pi/2$, which gives, of course, $F = 0$, and

$$A_x + B_x = W, \quad A_y + B_y = 0, \quad A_z + B_z = 0,$$

$$A_y - B_y = -\frac{b}{a} W \sin \phi, \quad A_z - B_z = -\frac{b}{a} W \cos \phi;$$

as ϕ may be assumed $= 0$ in this case, we find

$$A_y = -B_y = 0, \quad A_z = -B_z = -\frac{b}{2a} W.$$

The signs indicate that the upper hinge A is pulled out while the lower one B is pressed in.

2. CONSTRAINTS.

218. It has been shown in Art. 213 that the number of the conditions of equilibrium is six, for a rigid body that is perfectly free. This number will be diminished whenever the body is subject to conditions restricting its possible motions. Such

conditions, or **constraints**, may be of various kinds; the body may have a fixed point, or a fixed axis, or one of its points may be constrained to move along a given curve or to remain on a given surface, etc.

As explained in *Kinematics*, Art. 37, a free rigid body is said to have six degrees of freedom. The most general form of motion that it can have is a screw-motion, or twist, consisting of a rotation about a certain axis, and a translation along this axis; each of these resolves itself analytically into three rectangular components, and these six components may be regarded as constituting the six possible motions of the body, on account of which it is said to have six degrees of freedom.

Equilibrium will exist only when these six possible motions are prevented; hence there must be six conditions of equilibrium.

219. We proceed to consider some forms of constraint and the corresponding changes in the equations of equilibrium.

It is generally convenient in dynamics to replace such restraining conditions by forces, usually called **reactions**. Whenever it is possible to introduce such forces having the same effect as the given conditions, the body may be regarded as free, and the general equations of equilibrium can be applied.

Before considering the constraints of a rigid body, those of a single particle, or point, must be briefly discussed.

220. Particle constrained to a Surface. A free particle has three degrees of freedom; and accordingly its equilibrium is determined by three conditions (Art. 101):

$$\Sigma X=0, \quad \Sigma Y=0, \quad \Sigma Z=0. \quad (1)$$

If the co-ordinates determining the position of the particle be subject to *one condition*, expressed by an equation between these co-ordinates, the particle is said to have two degrees of freedom and one constraint. Its motion is restricted to the surface represented by the equation between its co-ordinates, say

$$\phi(x, y, z) = 0. \quad (2)$$

The condition that the particle should remain on this surface can be replaced by introducing the *reaction of the surface*, i.e. a force that is always so directed as not to allow the particle to leave the surface. Combining this force with the given forces acting on the particle, this particle can be regarded as free, and the general conditions of equilibrium must hold.

221. If the surface be *smooth*, i.e. if the particle move along it without friction, the reaction of the surface must be directed along the normal to the surface (2). Let N denote this normal reaction; N_x, N_y, N_z its components; then the conditions of equilibrium are

$$\Sigma X + N_x = 0, \quad \Sigma Y + N_y = 0, \quad \Sigma Z + N_z = 0. \quad (3)$$

The condition that N has the direction of the normal is expressed by the relations

$$\frac{N_x}{\phi_x} = \frac{N_y}{\phi_y} = \frac{N_z}{\phi_z} = \frac{N}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}, \quad (4)$$

where $\phi_x = \frac{\partial \phi}{\partial x}$, $\phi_y = \frac{\partial \phi}{\partial y}$, $\phi_z = \frac{\partial \phi}{\partial z}$ are obtained from (2).

Eliminating the reactions by means of (3), we find the *two conditions of equilibrium*,

$$\frac{\Sigma X}{\phi_x} = \frac{\Sigma Y}{\phi_y} = \frac{\Sigma Z}{\phi_z}. \quad (5)$$

The meaning of these equations is obvious; they express that the resultant of the given forces must have the direction of the normal to the surface.

The problem generally consists in finding the positions of equilibrium of the particle on the surface. The two equations (5) represent a curve whose intersections with the surface (2) give the required positions.

The magnitude of the reaction N is found from (3) :

$$N = \sqrt{(\sum X)^2 + (\sum Y)^2 + (\sum Z)^2}.$$

222. If the surface be *rough*, the total reaction of the surface lies within the cone of friction (Art. 187), and the resultant R of all other forces acting on the particle must therefore also fall within this cone.

The boundaries of the regions on the surface within which equilibrium is possible are found by considering the total reaction in its limiting position, *i.e.* when it makes the friction angle $\tan^{-1} \mu$ with the normal to the surface.

Let N represent the normal component of the total reaction ; N_x, N_y, N_z its components ; the force of friction μN lies in the tangent plane, and has, therefore, the components $\mu N dx/ds, \mu N dy/ds, \mu N dz/ds$, for motion along any curve s on the surface (2). These components of the friction must be given the double sign $\mp$, because the force of friction may act in either sense along the curve s . Thus, the conditions of equilibrium are

$$\begin{aligned}\sum X + N_x \mp \mu N \frac{dx}{ds} &= 0, \\ \sum Y + N_y \mp \mu N \frac{dy}{ds} &= 0, \\ \sum Z + N_z \mp \mu N \frac{dz}{ds} &= 0.\end{aligned}\tag{6}$$

To eliminate the reactions, multiply these equations by ϕ_x, ϕ_y, ϕ_z and add ; this gives

$$\sum X \cdot \phi_x + \sum Y \cdot \phi_y + \sum Z \cdot \phi_z + N_x \phi_x + N_y \phi_y + N_z \phi_z = 0,$$

since the differentiation of the equation of the surface (2) gives $\phi_x dx + \phi_y dy + \phi_z dz = 0$. Substituting for N_x, N_y, N_z their values from (4), the equation becomes

$$\sum X \cdot \phi_x + \sum Y \cdot \phi_y + \sum Z \cdot \phi_z + N \sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2} = 0.\tag{7}$$

This equation determines the normal reaction N of the surface.

To obtain an expression for μN , multiply the second of the equations (6) by ϕ_x , the third by ϕ_y and subtract; owing to the relations (4) this gives

$$\Sigma Y \cdot \phi_x - \Sigma Z \cdot \phi_y = \pm \mu N \left(\phi_x \frac{dy}{ds} - \phi_y \frac{dx}{ds} \right).$$

Similarly, we find

$$\Sigma Z \cdot \phi_x - \Sigma X \cdot \phi_z = \pm \mu N \left(\phi_x \frac{dz}{ds} - \phi_z \frac{dx}{ds} \right),$$

$$\Sigma X \cdot \phi_y - \Sigma Y \cdot \phi_z = \pm \mu N \left(\phi_y \frac{dx}{ds} - \phi_z \frac{dy}{ds} \right).$$

The left-hand members as well as the parentheses on the right are determinants of the second order; hence, squaring and adding, we find

$$\begin{aligned} & [(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2](\phi_x^2 + \phi_y^2 + \phi_z^2) \\ & - (\Sigma X \cdot \phi_x + \Sigma Y \cdot \phi_y + \Sigma Z \cdot \phi_z)^2 \\ & = \mu^2 N^2 (\phi_x^2 + \phi_y^2 + \phi_z^2). \end{aligned} \quad (8)$$

If N be now eliminated between (7) and (8), we find the *final condition of equilibrium* that must be fulfilled by the given forces independently of the reaction of the surface:

$$\begin{aligned} & [(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2](\phi_x^2 + \phi_y^2 + \phi_z^2) \\ & = (1 + \mu^2)(\Sigma X \cdot \phi_x + \Sigma Y \cdot \phi_y + \Sigma Z \cdot \phi_z)^2. \end{aligned} \quad (9)$$

Putting this equation into the form

$$\frac{1}{\sqrt{1 + \mu^2}} = \frac{\Sigma X}{R} \cdot \frac{\phi_x}{R'} + \frac{\Sigma Y}{R} \cdot \frac{\phi_y}{R'} + \frac{\Sigma Z}{R} \cdot \frac{\phi_z}{R'},$$

where $R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2$ and $R'^2 = \phi_x^2 + \phi_y^2 + \phi_z^2$, it is seen to express the fact that the resultant R of the given forces makes the friction angle ϕ with the normal, each member of the equation being an expression for the cosine of this angle.

The regions of the surface (2) on which the particle is in equilibrium are cut out of this surface by the surface (9).

223. Particle constrained to a Curve. If the particle be subject to two conditions,

$$\phi(x, y, z) = 0, \quad \psi(x, y, z) = 0, \quad (10)$$

so that it has two constraints and but one degree of freedom, its motion is restricted to the curve of intersection of the two surfaces (10). The particle may be imagined as a small sphere moving within a tube, or as a small ring or bead sliding along a thin wire.

224. Let the curve be *smooth*, so that its total reaction is along the normal to the curve. Denoting this normal reaction again by N , its components by N_x, N_y, N_z , the conditions of equilibrium are

$$\Sigma X + N_x = 0, \quad \Sigma Y + N_y = 0, \quad \Sigma Z + N_z = 0, \quad (11)$$

$$\text{or} \quad \frac{\Sigma X}{N_x} = \frac{\Sigma Y}{N_y} = \frac{\Sigma Z}{N_z} \quad (12)$$

The condition that N has the direction of the normal can be expressed in the form

$$N_x dx + N_y dy + N_z dz = 0,$$

which, by (12), reduces to

$$\Sigma X \cdot dx + \Sigma Y \cdot dy + \Sigma Z \cdot dz = 0.$$

Differentiating the equations of the curve (10), we find

$$\phi_x dx + \phi_y dy + \phi_z dz = 0,$$

$$\psi_x dx + \psi_y dy + \psi_z dz = 0;$$

and eliminating the differentials between the last three equations, the single condition of equilibrium, independent of the reactions, is found in the form

$$\begin{vmatrix} \Sigma X & \Sigma Y & \Sigma Z \\ \phi_x & \phi_y & \phi_z \\ \psi_x & \psi_y & \psi_z \end{vmatrix} = 0. \quad (13)$$

The intersections of the surface (13) with the curve (10) give the positions of equilibrium of the particle on the curve.

The reaction of the curve, or the pressure on the curve which is equal and opposite to this reaction, can then be found from the equations (11).

225. For a *rough* curve, the total reaction resolves itself into a normal component N and a tangential component μN , which represents the frictional resistance. The equations of equilibrium are

$$\begin{aligned} \Sigma X + N_x \mp \mu N \frac{dx}{ds} &= 0, \\ \Sigma Y + N_y \mp \mu N \frac{dy}{ds} &= 0, \\ \Sigma Z + N_z \mp \mu N \frac{dz}{ds} &= 0. \end{aligned} \quad (14)$$

Transposing the third terms, multiplying by dx/ds , dy/ds , dz/ds , and adding, we find, since $N_x dx + N_y dy + N_z dz = 0$,

$$\Sigma X \cdot \frac{dx}{ds} + \Sigma Y \cdot \frac{dy}{ds} + \Sigma Z \cdot \frac{dz}{ds} = \pm \mu N. \quad (15)$$

Multiplying the second of the equations (14) by dz/ds , the third by dy/ds , and subtracting, we have

$$\Sigma Y \cdot \frac{dz}{ds} - \Sigma Z \cdot \frac{dy}{ds} = - \left(N_y \frac{dz}{ds} - N_z \frac{dy}{ds} \right);$$

similarly,
$$\Sigma Z \cdot \frac{dx}{ds} - \Sigma X \cdot \frac{dz}{ds} = - \left(N_z \frac{dx}{ds} - N_x \frac{dz}{ds} \right),$$

$$\Sigma X \cdot \frac{dy}{ds} - \Sigma Y \cdot \frac{dx}{ds} = - \left(N_x \frac{dy}{ds} - N_y \frac{dx}{ds} \right).$$

Each member being a determinant of the second order, we find by squaring and adding the three equations,

$$(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 - \left(\Sigma X \cdot \frac{dx}{ds} + \Sigma Y \cdot \frac{dy}{ds} + \Sigma Z \cdot \frac{dz}{ds} \right)^2 = N^2. \quad (16)$$

The reaction N can now be eliminated between (15) and (16), and we obtain the single condition of equilibrium independent of the reaction :

$$\Sigma X \cdot \frac{dx}{ds} + \Sigma Y \cdot \frac{dy}{ds} + \Sigma Z \cdot \frac{dz}{ds} = \pm \frac{\mu}{\sqrt{1+\mu^2}} \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}. \quad (17)$$

The differential coefficients dx/ds , dy/ds , dz/ds must satisfy the differential equations of the curve (10), viz. :

$$\phi_x \frac{dx}{ds} + \phi_y \frac{dy}{ds} + \phi_z \frac{dz}{ds} = 0,$$

$$\psi_x \frac{dx}{ds} + \psi_y \frac{dy}{ds} + \psi_z \frac{dz}{ds} = 0.$$

If the values of dx/ds , dy/ds , dz/ds be determined from the last three equations and substituted into the relation

$$\left(\frac{dx}{ds} \right)^2 + \left(\frac{dy}{ds} \right)^2 + \left(\frac{dz}{ds} \right)^2 = 1,$$

the equation of a surface will result, which cuts out, on the curve (10), the limits between which equilibrium is possible.

226. Rigid Body with a Fixed Point. A body that is free to turn about a fixed point A can be regarded as free if the reaction A of this point be introduced and combined with the other forces acting on the body.

Let A_x , A_y , A_z be the components of A ; then, taking the fixed point A as origin, the six equations of equilibrium (Art. 213) are

$$\Sigma X + A_x = 0, \quad \Sigma Y + A_y = 0, \quad \Sigma Z + A_z = 0,$$

$$\Sigma (yZ - zY) = 0, \quad \Sigma (zX - xZ) = 0, \quad \Sigma (xY - yX) = 0.$$

The first three of these equations serve to determine the reaction of the fixed point; the last three are the actual conditions of equilibrium corresponding to the three degrees of freedom of a body with a fixed point.

Hence, *a rigid body having a fixed point is in equilibrium if the sum of the moments of all the forces vanishes for any three axes passing through the fixed point and not situated in the same plane.*

227. Rigid Body with a Fixed Axis. A body with a fixed axis has but one degree of freedom; indeed, the only possible motion consists in rotation about this axis.

An axis is fixed as soon as two of its points, say A, B , are fixed. Hence, introducing the reactions $A_x, A_y, A_z, B_x, B_y, B_z$ of these points, the body can be regarded as free. If the point B be taken as origin, the line BA as axis of z (Fig. 73), the equations of equilibrium become

$$\begin{aligned}\Sigma X + A_x + B_x &= 0, & \Sigma Y + A_y + B_y &= 0, & \Sigma Z + A_z + B_z &= 0, \\ \Sigma (yZ - zY) - B_y a &= 0, & \Sigma (zX - xZ) - B_x a &= 0, & \Sigma (xY - yX) &= 0,\end{aligned}$$

where $a = BA$.

The last of the six equations is the only independent condition of equilibrium of the constrained body; the first five determine $A_x, B_x, A_y, B_y, A_z + B_z$. The two z -components cannot be found separately, since they act in the same straight line.

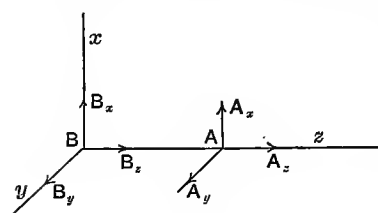


Fig. 73.

Hence, *a rigid body having a fixed axis is in equilibrium if the sum of the moments of all the forces vanishes for the fixed axis.*

228. If, in the preceding article, the axis be not absolutely fixed, but only fixed in direction so that *the body can rotate about the axis and also slide along it*, we have evidently

$$A_z=0, \quad B_z=0;$$

hence, by the third equation of equilibrium,

$$\Sigma Z=0,$$

as an additional condition of equilibrium.

The body has in this case two degrees of freedom.

229. Rigid Body with a Fixed Plane. A body constrained to slide on a fixed plane has three degrees of freedom. At every point of contact between the body and the plane, the latter exerts a reaction. As all these reactions are parallel, they can be combined into a single resultant N . Taking the fixed plane as the plane xy , N will be parallel to the axis of z ; hence, if a , b , 0 be the co-ordinates of its point of application, the six equations of equilibrium are

$$\Sigma X=0, \quad \Sigma Y=0, \quad \Sigma Z+N=0,$$

$$\Sigma(yZ-zY)+bN=0, \quad \Sigma(zX-xZ)-aN=0, \quad \Sigma(xY-yX)=0.$$

The third, fourth, and fifth equations determine the reaction N and the co-ordinates a , b of its point of application. The three other equations are the actual conditions of equilibrium; they agree, of course, with the three conditions of equilibrium of a plane system as found in Art. 143.

If there be not more than three points of contact (or supports) between the body and the fixed plane, the reactions of these points can be found separately. Let A_1 , A_2 , A_3 be the three points of contact; N_1 , N_2 , N_3 the required reactions; a_1 , b_1 , a_2 , b_2 , a_3 , b_3 the co-ordinates of A_1 , A_2 , A_3 ; then N must be resolved into three parallel forces passing through these points, and the conditions are

$$N_1+N_2+N_3=N,$$

$$a_1N_1+a_2N_2+a_3N_3=aN,$$

$$b_1N_1+b_2N_2+b_3N_3=bN.$$

These three equations determine N_1, N_2, N_3 , unless the three points A_1, A_2, A_3 be situated in a straight line; for in this case the determinant of the coefficients of N_1, N_2, N_3 vanishes,

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

The reactions become almost indeterminate whenever there are more than three points of contact.

230. In addition to the works of ROUTH, MINCHIN, PRICE, TODHUNTER, mentioned in Art. 199, the student is referred, in particular for the more advanced parts of the subject, to W. SCHELL, *Theorie der Bewegung und der Kräfte*, Vol. II., Leipzig, Teubner, 1880; A. F. MÖBIUS, *Lehrbuch der Statik*, Leipzig, Göschen, 1837, reprinted in MÖBIUS's *Gesammelte Werke*, Vol. III., Leipzig, Hirzel, 1886; MOIGNO, *Statique*, Paris, Gauthier-Villars, 1868; J. SOMOFF, *Theoretische Mechanik*, übersetzt von A. Ziwet, Vol. II., Leipzig, Teubner, 1879; E. COLLIGNON, *Statique*, Paris, Hachette, 1889; THOMSON AND TAIT *Natural Philosophy*, Part. II., Cambridge, University Press, 1890.

VII. *The Principle of Virtual Work.*

231. Work has been defined in Art. 72 as the product of a force into the displacement of its point of application in the direction of the force.

Thus the expansive force F of the steam in the cylinder of a steam-engine, in pushing the piston through a distance s , is said to *do work*, and this work is measured by the product Fs . Similarly the force of gravity, *i.e.* the attractive force of the earth's mass, does work on a falling body.

The resistance to be overcome by the engine, in the former case, and the resistance of the air in the latter, are also forces acting on the body during its displacement. But as the sense of the displacement is opposite to that of these forces, their work is negative; *work is done against these forces*. Thus the muscular force of a man who raises a weight does work against gravity; if the weight he holds is so heavy as to pull him down, gravity does work against his force; if he merely tugs at a weight without being able to lift it, the work is zero, because the displacement is zero.

232. In general, the point of application of a force F will be acted upon by a number of different forces, so that the displacement s of this point will not necessarily take place in the direction of F . In this general case *the work of a force is defined as the product of the force into the projection of the displacement of its point of application on the direction of the force*.

In Fig. 74, for instance, the particle P while acted upon by the force F (and any number of other forces) is displaced from P to P' ; hence if $PP' = s$, and $\angle P'PQ = \phi$, the work of the force F is

$$W = Fs \cos \phi. \quad (1)$$

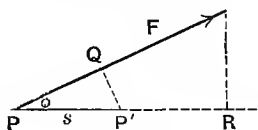


Fig. 74.

It is obvious that this work might also be defined as the

product of the displacement into the projection of the force on the displacement ; for we have

$$Fs \cos \phi = F \cdot PQ = s \cdot PR.$$

The work of a force is evidently positive or negative according as the angle ϕ is less or greater than $\pi/2$, provided we select for ϕ always that angle between F and s which is not greater than P .

233. The above definition of work assumes that the force F remains constant, both in magnitude and direction, while displacement s takes place, and that this displacement is rectilinear. If either, or both, of these conditions be not fulfilled, the definition can be applied only to infinitesimal displacements ds . As the work done by a finite force F during such a displacement ds is infinitesimal, we have

$$dW = Fds \cos \phi ; \quad (2)$$

and the total work done by any variable force F while its point of application is displaced along any straight or curvilinear path PQ , is obtained by integrating from P to Q :

$$W = \int_P^Q F \cos \phi \, ds. \quad (3)$$

234. Since work can always be regarded as the product of a force into a length, its dimensions are found by multiplying those of force, $\mathbf{MLT}^{-2}$ (Art. 64), by $\mathbf{L}$; hence, the *dimensions of work* are

$$\mathbf{W} = \mathbf{ML}^2\mathbf{T}^{-2}.$$

The unit of work is the work of a unit force (poundal, dyne) through a unit distance (foot, centimetre). The unit of work in the F.P.S. system is called the **foot-poundal**; in the C.G.S. system, the **erg**. Thus, the erg is the amount of work done by a force of one dyne acting through a distance of one centimetre. These are the *absolute* units.

In the gravitation system where the pound, or the kilogramme, is taken as unit of force, the British unit of work is the **foot-pound**, while in the metric system it is customary to use the **kilogramme-metre** as unit.

235. The numerical relations between these units are obtained as follows. Let x be the number of ergs in the foot-poundal, then (comp. Art. 66),

$$x \cdot \frac{\text{gm. cm.}^2}{\text{sec.}^2} = 1 \cdot \frac{\text{lb. ft.}^2}{\text{sec.}^2},$$

hence
$$x = \frac{\text{lb.}}{\text{gm.}} \cdot \left(\frac{\text{ft.}}{\text{cm.}} \right)^2 = 453.59 \times 30.4797^2 = 4.2139 \times 10^5;$$

i.e. 1 foot-poundal = 4.2139×10^5 ergs, and 1 erg = 2.3721×10^{-6} = 0.000 002 372 1 foot-poundal.

Again, let x be the number of kilogramme-metres in 1 foot-pound, then

$$x \text{ kg. m.} = 1 \text{ ft. lb.},$$

hence
$$x = \frac{\text{lb.}}{\text{kg.}} \cdot \frac{\text{ft.}}{\text{m.}} = 0.45359 \times 0.3048 = 0.13825,$$

i.e. 1 foot-pound = 0.13825 kilogramme-metres.

Finally, 1 foot-pound = g foot-poundals (Art. 69); hence 1 foot-pound = 1.356×10^7 ergs, and 1 erg = 7.3737×10^{-8} foot-pounds, if $g = 32.2$.

236. Exercises.

(1) A *joule* being defined as 10^7 ergs, shows that 1 foot-pound = 1.356 joules, and that 1 joule is about $3/4$ foot-pound.

(2) Show that a kilogramme-metre is nearly 10^8 ergs.

(3) What is the work done against gravity in raising 300 lbs. through a height of 25 ft. : (a) in foot-pounds, (b) in ergs?

(4) Find the work done against friction in moving a car weighing 3 tons through a distance of fifty yards on a level road, the coefficient of friction being 0.02.

(5) A mass of 12 lbs. slides down a smooth plane inclined at an angle of 30° to the horizon, through a distance of 25 ft.; what is the work done by gravity?

237. It follows from the definition of work that, if any number of forces $F_1, F_2, \dots, F_n$ act on a particle P , *the sum of their works for any displacement $PP' = ds$ is equal to the work of their resultant R for the same displacement.* For, the resultant R being the closing line of the polygon constructed by adding the forces $F_1, F_2, \dots, F_n$ geometrically, the projection of R on any direction, such as PP' , is equal to the sum of the projections of the forces F on the same line (Art. 89); that is, if $\alpha_1, \alpha_2, \dots, \alpha_n$ be the angles made by $F_1, F_2, \dots, F_n$ with PP' , and α the angle between R and PP' , we have

$$F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \dots + F_n \cos \alpha_n = R \cos \alpha;$$

multiplying this equation by ds , we obtain the above proposition

$$F_1 \cos \alpha_1 ds + F_2 \cos \alpha_2 ds + \dots + F_n \cos \alpha_n ds = R \cos \alpha ds,$$

which expresses the so-called **principle of work** for a single particle.

238. When the particle is in equilibrium, so that the forces do not actually change the motion, we may derive from this proposition a convenient expression for the conditions of equilibrium by considering displacements that *might be* given to the particle. Such displacements are called *virtual*, and the corresponding work of any of the forces is called **virtual work**.

It is customary to denote a virtual displacement by δs , the letter δ being used to distinguish from an actual displacement ds ; this distinction becomes of importance in kinetics.

239. The resultant being zero in the case of equilibrium, *the sum of the virtual works of all forces acting on the particle must be zero for any virtual displacement, i.e.*

$$F_1 \cos \alpha_1 \delta s + F_2 \cos \alpha_2 \delta s + \dots + F_n \cos \alpha_n \delta s = 0. \quad (4)$$

As the resultant must vanish if its three projections vanish for any three axes not lying in the same plane, the necessary

and sufficient conditions of equilibrium of a single particle are that *the sum of the virtual works of all forces must be zero for any three virtual displacements not all in the same plane.*

This is the **principle of virtual work** for a single particle.

If the particle be referred to a rectangular system of co-ordinates, its displacement δs can be resolved into three component displacements δx , δy , δz , parallel to the axes. The forces acting on the particle being replaced by their components X , Y , Z , the sum of their virtual works, for the displacement δs is $\Sigma X \cdot \delta x + \Sigma Y \cdot \delta y + \Sigma Z \cdot \delta z$. Hence the analytical expression for the principle of virtual work :

$$\Sigma X \cdot \delta x + \Sigma Y \cdot \delta y + \Sigma Z \cdot \delta z = 0. \quad (5)$$

As the displacements δx , δy , δz are independent of each other, and perfectly arbitrary, this single equation is equivalent to the three equations

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

which are the ordinary conditions of equilibrium of a single particle.

240. The principle of virtual work is particularly useful in eliminating the unknown reactions arising from constraints.

Suppose the particle be constrained to a smooth surface or curve. After introducing the normal reaction of the surface or curve the particle can be regarded as free ; and the equation of virtual work can be used to express the conditions of equilibrium. This equation will, in general, contain the unknown reaction. But as this reaction has the direction of the normal, it will be eliminated if the virtual displacement be selected along a tangent. Hence, *in the case of constraintment to a surface, the two conditions of equilibrium independent of the reaction are found by forming the equation of virtual work for virtual displacements along any two tangents to the surface ; and in the case of constraintment to a curve the one such condition is found from a virtual displacement along the tangent.*

If it be required to find the normal pressure on the surface

or curve, which is of course equal and opposite to the reaction, it can be found from a virtual displacement along the normal.

241. If the equation (4) which expresses the principle of virtual work be divided by the element of time δt , during which the displacement δs would take place, the factor $\delta s/\delta t = v$ represents a *virtual velocity*, and the equation becomes

$$F_1 \cos \alpha_1 \cdot v + F_2 \cos \alpha_2 \cdot v + \dots + F_n \cos \alpha_n \cdot v = 0.$$

On account of this form, the proposition is often called the *principle of virtual velocities*.

The product of a force into the virtual velocity of its point of application in the direction of the force, $F \cos \alpha \cdot v$, is sometimes called the *virtual moment* of the force.

242. The principle of virtual work can readily be extended to the case of a rigid body acted upon by any number of forces.

The forces acting on a rigid body can always be reduced to a resultant R and a resulting couple H (Art. 200). This reduction is based on the supposition (Art. 84) that the point of application of a force can be displaced arbitrarily along the line

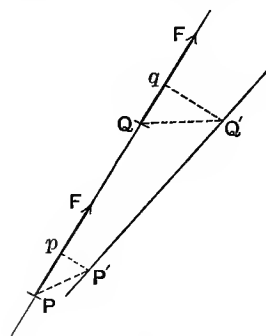


Fig. 75.

of the force. It can be shown that such a displacement of the point of application P of a force F (Fig. 75), from P to Q along the line of the force, does not affect the work done by the force in any infinitesimal displacement of the body. Let $PP' = \delta s$ be the displacement of P , $QQ' = \delta s'$ that of Q ; let p and q be the projections of P and Q on the line of the force F ; then, since the body is rigid, $P'Q' = PQ$; and consequently Qq will differ from Pp only by

an infinitesimal of an order higher than the order of the displacement $PP' = \delta s$. Hence,

$$F \cdot Pp = F \cdot Qq.$$

It may here be noted that, in general, the principle of virtual work must be understood to mean that the sum of the works of the forces differs from the work of their resultant by an infinitesimal of an order higher than that of the virtual displacement. It does not mean that the difference is absolutely zero.

243. Owing to the proposition proved in the preceding article, the sum of the works of all the forces acting on a rigid body is equal to the sum of the works of the resultant R and the resulting couple H for any infinitesimal displacement of the body, and the work of the forces is not changed by such a displacement.

It follows that the necessary and sufficient conditions of equilibrium of a rigid body (Art. 201), viz.

$$R=0, H=0,$$

can be expressed by saying that *the sum of the virtual works of all the forces must be zero for any infinitesimal displacement of the body.*

For when the forces are in equilibrium, this condition is evidently fulfilled. To prove that there must be equilibrium whenever this condition is fulfilled, it is only necessary to show that both R and H must vanish if the sum of their works is zero for any infinitesimal displacement.

To see this, consider first a displacement of translation, δs , parallel to R . The work of R will be $R\delta s$ while the works of the two forces constituting H are equal and opposite, so that the work of H is zero. As the sum of the works of R and H must vanish by hypothesis, it follows that $R=0$.

Next consider a displacement of rotation $\delta\theta$ about an axis parallel to the vector H . Taking this axis so as to intersect R and bisect the arm p of the couple H , the work of R will be zero while that of each of the forces F of the couple H will be $\frac{1}{2}p\delta\theta \cdot F$; hence the whole work of H is $Fp\delta\theta = H\delta\theta$. As

the sum of the works of R and H must vanish by hypothesis, it follows that $H=0$.

The two conditions $R=0$, $H=0$, are, therefore, both fulfilled.

244. The following examples may serve to illustrate the application of the principle of virtual work.

To find the force just necessary to move a cylinder of radius r and weight W up a plane inclined at an angle α to the horizon by means of a crow-bar of length l set at an angle β to the horizon (Fig. 76).

Let s be the distance from the fulcrum A of the crow-bar to the point of contact B of the cylinder with the plane.

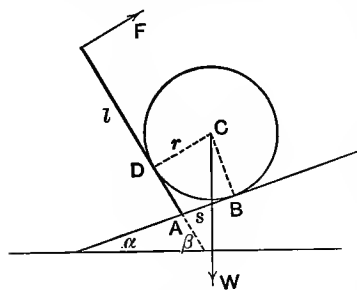


Fig. 76.

Turning the crow-bar about A by an angle $\delta\beta$, the work of the force F acting at the end of the bar is $F \cdot l \delta\beta$. The corresponding displacement of the centre C of the circle, which is the point of application of the force W , is parallel to the inclined plane, and may be regarded as the differential δs of the distance $AB = s$. The

work of W is, therefore, $W \delta s \sin \alpha$. This gives the equation of work

$$F \cdot l \delta\beta = W \cdot \delta s \sin \alpha;$$

hence

$$F = \frac{W}{l} \sin \alpha \cdot \frac{\delta s}{\delta\beta}.$$

The relation between s and β can be found by projecting $ABCD A$ on the vertical line; this gives

$$r \cos \alpha + s \sin \alpha = r \cos \beta + s \sin \beta,$$

whence

$$s = r \frac{\cos \beta - \cos \alpha}{\sin \alpha - \sin \beta}.$$

Differentiating the former equation, we find

$$\sin \alpha \delta s = -r \sin \beta \delta\beta + s \cos \beta \delta\beta + \sin \beta \delta s,$$

$$\therefore \frac{\delta s}{\delta\beta} = \frac{s \cos \beta - r \sin \beta}{\sin \alpha - \sin \beta} = r \frac{\cos^2 \beta - \cos \alpha \cos \beta - \sin \alpha \sin \beta + \sin^2 \beta}{(\sin \alpha - \sin \beta)^2},$$

or
$$\frac{1}{r} \frac{\delta s}{\delta \beta} = \frac{1 - \cos(\alpha - \beta)}{(\sin \alpha - \sin \beta)^2} = \frac{1}{1 + \cos(\alpha + \beta)}$$

Hence, finally,
$$F = \frac{r}{l} \frac{\sin \alpha}{1 + \cos(\alpha + \beta)} \cdot W.$$

245. A weightless rod of length $AB=1$ rests at C on a horizontal cylinder whose axis is at right angles to the vertical plane through the rod; its lower end A leans against a vertical wall, and from its upper end B a weight W is suspended. Determine the reactions at A and C , and the distance $AC = x$ for equilibrium, if the distance $CD = a$ of the point of support from the vertical wall is given (Fig. 75).

(a) Let A glide vertically upwards, C remaining in contact. At A as well as at C the forces are perpendicular to the displacements; hence, putting $\delta B = y$, we have $W\delta y = 0$.

$$\therefore \delta y = 0.$$

Also,
$$\frac{y}{l-x} = \frac{\sqrt{x^2 - a^2}}{x}.$$

$$\therefore \delta y = \delta \left\{ \left(\frac{l}{x} - 1 \right) \sqrt{x^2 - a^2} \right\} = 0,$$

whence, $(l-x)x^2 - l(x^2 - a^2) = 0$,

or
$$x^3 = a^2 l.$$

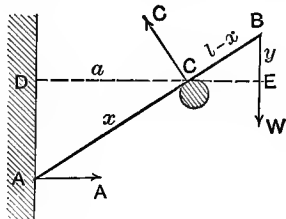


Fig. 77.

(b) Give the rod a vertical displacement to a parallel position :

$$-W\delta y + C \frac{\pi}{x} \delta y = 0; \therefore C = \frac{x}{a} W = \sqrt[3]{\frac{l}{a}} \cdot W.$$

(c) Give the rod a displacement in its own direction :

$$A \frac{a}{x} \delta s + C \cos \frac{\pi}{2} \cdot \delta s - W \frac{\sqrt{x^2 - a^2}}{x} \cdot \delta s = 0.$$

$$\therefore A = \frac{\sqrt{x^2 - a^2}}{a} \cdot W = \sqrt{\left(\frac{l}{a} \right)^{\frac{2}{3}} - 1} \cdot W.$$

246. In a parallelogram formed by four rods with hinges at the vertices, elastic strings are stretched along the diagonals. Determine the ratio of the tensions in these strings.

Let m, m' be the lengths of the diagonals, T, T' the tensions, and $\delta m, \delta m'$ the changes of length of the diagonals when the parallelogram is slightly deformed; then by the principle of virtual work

$$T \delta m + T' \delta m' = 0. \quad (1)$$

From geometry we have; if a, b are the sides of the parallelogram,

$$m^2 + m'^2 = 2a^2 + 2b^2,$$

hence, differentiating, $m \delta m + m' \delta m' = 0. \quad (2)$

From (1) and (2) we find

$$T/T' = m/m'. \quad (3)$$

247. For the purposes of statics a body is regarded as rigid if the points of application of all the forces acting on the body have invariable distances from each other; these points may be imagined connected by a framework of rigid rods. The tensions in these connecting rods, since they occur in pairs of equal and opposite forces acting along rigid lines, do not enter into the equation of virtual work. One of the chief advantages of the principle of virtual work consists in this elimination of these *internal forces*.

Let us now generalize the idea of the rigid body by assuming the points of application of the forces to be connected by rods or threads which may even be elastic; the points may also be constrained to move on smooth surfaces or curves; friction, however, is to be excluded.

Let x, y, z be the co-ordinates of one of the points, P ; x', y', z' , those of another point, Q ; let l be the length of the connecting thread or rod, PQ ; α, β, γ , its direction cosines; and let T be the tension or stress in PQ . If the whole system be subjected to any infinitesimal displacement, for which $\delta x, \delta y, \delta z$ are the component displacements of P , $\delta x', \delta y', \delta z'$ those of Q , the sum of the works of the two equal and opposite forces T for this displacement will be

$$T\alpha \delta x + T\beta \delta y + T\gamma \delta z - T\alpha \delta x' - T\beta \delta y' - T\gamma \delta z',$$

or, since $\alpha = (x - x')/l$, $\beta = (y - y')/l$, $\gamma = (z - z')/l$,

$$\frac{T}{l} [(x - x')(\delta x - \delta x') + (y - y')(\delta y - \delta y') + (z - z')(\delta z - \delta z')].$$

Differentiating the relation

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = l^2,$$

we have

$$(x - x')(\delta x - \delta x') + (y - y')(\delta y - \delta y') + (z - z')(\delta z - \delta z') = l \delta l;$$

hence the sum of the virtual works of the two tensions T reduces to

$$T \delta l,$$

which is of course zero when the connecting rod is rigid.

It thus appears that the internal reactions of a system of points connected as above described, are eliminated from the equation of virtual work by selecting the virtual displacements so as to leave the lengths of the connecting rods or threads unchanged. This can always be done when the rods and threads are not elastic. When they are elastic, the equation of virtual work will contain terms of the form $T \delta l$. These terms must then be determined from the known relation between the tension and the length of an elastic rod or thread.

248. It is somewhat difficult to prove the principle of virtual work for the most general case of any system of bodies although this is the case in which it finds its most important application. It is evident, however, that the principle will be true in this general case provided that all the connections and reactions between the different bodies constituting the system be expressed by means of forces and introduced into the equation of virtual work. The difficulty lies in expressing the connections existing between the parts of the system by means of forces.

But most of these internal reactions can be shown to disappear from the equation of virtual work, so that they need not be taken into account.

Thus the tension of an inelastic thread or rod being composed of two equal and opposite forces does no work in any infinitesimal displacement; the work of the reaction of a fixed point or fixed axis is zero, because the point of application cannot move in the direction of the force; the normal reaction of a surface along which a body is constrained to slide does no work, because any possible displacement is at right angles to the force. If, however, the surface be rough, the friction will in general do work.

249. A full discussion of the principle of virtual work for the most general case cannot be given in this place. It must suffice to state it and to illustrate its application in a few special cases.

The necessary and sufficient condition of equilibrium of a system of bodies the connections between which can be expressed by forces is this, that the sum of the virtual works of all the forces must vanish for all displacements consistent with the geometrical conditions to which the system may be subject.

The internal reactions between the different parts of the system, with the exception of friction and elasticity, will in general not enter into the equation of virtual work. Such reactions can therefore be neglected while friction and elastic tensions must be included among the forces acting on the system.

250. Let P be one of the forces, δp the projection on its direction of the virtual displacement of its point of application; then the principle of virtual work requires that

$$\Sigma P \delta p = 0.$$

If X, Y, Z be the rectangular components of P , and x, y, z the co-ordinates of its point of application, the same condition can be expressed in the form:

$$\Sigma (X \delta x + Y \delta y + Z \delta z) = 0.$$

251. Whether the system be in equilibrium or not, the quantity $\Sigma P\delta p = \Sigma(X\delta x + Y\delta y + Z\delta z)$ represents the element of work δW done by the forces in a virtual displacement. For an *actual* displacement of the system δ should be replaced by d , and we have

$$dW = \Sigma Pdp = \Sigma(Xdx + Ydy + Zdz)$$

for the work done in the infinitesimal displacement.

Most of the forces occurring in nature are such as to make this quantity an exact differential. In this case the forces are said to form a *conservative system*, and the equation can be integrated from any initial position or configuration of the system to any final position or configuration without reference to the intermediate positions.

Taking the initial position as the standard of reference, the result can be written in the form

$$W = \int Pdp = U - U_0,$$

where U is a function of the co-ordinates (or other quantities) determining the position and configuration of the system, while U_0 is the initial value of U .

Leaving the standard of reference indeterminate, the equation can be written in the form

$$W = U + C,$$

C being merely a symbol for the constant of integration. The function W is called the *work function* or *force function*.

If the final position of the system be taken as standard of reference, and U_1 be the value of U in the final position, the equation takes the form

$$V = U_1 - U,$$

where V is called the *potential energy* of the forces with reference to the final position.

252. If the work function

$$W = U + C$$

be given as a known function of the co-ordinates determining the configuration of the system, the positions of equilibrium of the system can be found from the condition

$$dW = 0,$$

which expresses that *the work function W is a maximum, or a minimum (or stationary)*. It can be shown without difficulty that the equilibrium is stable when the work function is a maximum, and unstable when the work function is a minimum.

As the potential energy by the formulæ of Art. 248 is equal to the work function, but of opposite sign, it follows that the equilibrium is stable or unstable according as the potential energy is a minimum or a maximum.

253. The special case when the only forces acting are the weights of the particles constituting the system is worth mentioning.

Let m be the mass of one of the particles, mg its weight, and z its height above a horizontal plane of reference. Then the virtual work of the weights is

$$\delta W = - \sum mg \delta z = - g \sum m \delta z.$$

If $\bar{z}$ be the height of the centroid of the system above the plane of reference, we have $\sum mz = \sum m \cdot \bar{z}$; hence $\sum m \delta z = \sum m \cdot \delta \bar{z}$. The work function is, therefore,

$$W = - g \sum m \cdot \bar{z} + C,$$

and this becomes a maximum or minimum according as $\bar{z}$ is a minimum or maximum; *i.e. the equilibrium is stable or unstable according as the centroid of the system is at its least or greatest height.*

254. It is the object of every *machine* to do work in a certain prescribed way, *i.e.* to exert force, or overcome a resistance, through a certain distance. The various forces of nature, such as the muscular force of man and other animals, the force of gravity, the pressure of the wind, electricity, the expansive force of steam or gas, etc., are called upon for this purpose. In most cases it would not do to apply these forces

directly; they must be controlled, guided, and transformed in various ways to become useful, and this is done by interposing the machine between the given *driving force*, commonly called the *power*, and the force which is to do the final work, usually called the *resistance*, *load*, or *weight*. We shall in general denote the "power" by P , the "weight" by Q .

The term *power* is somewhat objectionable in this connection, being here used to denote a force, while in Kinetics it is used for the *rate of doing work*.

255. *The ratio Q/P of the weight to the power is called the mechanical advantage of the machine.*

Under the action of the power P its point of application as well as that of the weight Q is displaced. The corresponding work of the force P may be called the *available* or *total work*; that of the force Q is called the *useful work*.

The ratio of the useful work to the total work is called the **efficiency** of the machine.

In all machines this efficiency is a proper fraction, owing to the fact that the work done by P must balance not only the useful work, but also the so-called wasteful work due to friction, stiffness of ropes, slipping of belts, lack of rigidity, etc.

256. For a more complete discussion of the principle of virtual work the student is referred to MINCHIN'S *Statics*, Vol. I., pp. 78-96, 160-180, and Vol. II., pp. 98-188; ROUTH'S *Statics*, Vol. I., pp. 146-197; SCHELL'S *Theorie der Bewegung und der Kräfte*, Vol. II., pp. 166-211; and J. PETERSEN, *Statik fester Körper*, übersetzt von R. von Fischer-Benzon, Kopenhagen, Höst, 1882, pp. 114-124.

VIII. *Theory of Attractive Forces.*

I. ATTRACTION.

257. Among the various kinds of forces introduced in physics for describing and interpreting natural phenomena, forces of attraction and repulsion occupy a most prominent place.

According to *Newton's law* (the law of universal or cosmical gravitation, the law of nature), every particle of matter attracts every other such particle with a force proportional to the masses and inversely proportional to the square of the distance of the particles.

If m, m' be the masses of the two particles, r their distance, and κ a constant, the mathematical expression for the force of mutual attraction *exerted by each particle on the other, or for the stress between them, is, therefore,

$$F = \kappa \frac{mm'}{r^2}. \quad (1)$$

258. Each particle is here regarded as a mathematical point at which its mass is concentrated. The attractive force would, therefore, approach the limit ∞ as the distance between the points approaches the value 0. To prevent the introduction of infinite forces, we may in such limiting cases regard the particles as very small homogeneous spheres formed of an impenetrable substance. If r, r' be the radii, ρ, ρ' the densities of the spheres, the attraction reaches a finite maximum value $F_{\max}$, when the spheres are in contact, viz.

$$F_{\max} = \kappa \frac{\frac{4}{3}\pi\rho r^3 \cdot \frac{4}{3}\pi\rho r'^3}{(r+r')^2} = \frac{16}{9}\pi^2\kappa\rho\rho' \cdot \frac{r^3r'^3}{(r+r')^2},$$

which is very small of the fourth order if r, r' be very small of the first order. Thus, for $r=r'$,

$$F_{\max} = \frac{4}{9}\pi^2\kappa\rho\rho' \cdot r^4.$$

259. In many applications of the theory of attraction, in particular in electricity and magnetism, it is convenient to consider forces of *repulsion*. This only requires a change of sign in the expression for the force, and this sign may be regarded as attaching to the mass of one of the particles; in other words, if the mass of a centre of attraction be taken as positive, that of a centre of repulsion is taken to be negative, or *vice versa*.

260. While according to Newton's law (1) the force is inversely proportional to the square of the distance, it is often convenient to use forces depending upon the distance r in a different way. Thus the theory of Newtonian attraction can be generalized by assuming for the force between two particles m , m' the law

$$F = \kappa m m' f(r), \quad (2)$$

where $f(r)$ represents any function of the distance r .

When nothing is said to the contrary, we shall here always assume that $f(r) = 1/r^2$, as in Newton's law (1).

261. The constant κ evidently represents the force with which two particles, each of mass 1, attract each other when at the distance 1. It is a physical constant to be determined by experiment, and its numerical value depends on the units of measurement adopted. What can be directly observed is of course not the force itself, but the acceleration it produces. Dividing the force F , as given by formula (1), by the mass m' of the particle on which it acts, we find for the *acceleration* j produced by the attraction of the mass m in the mass m' at the distance r from m :

$$j = \kappa \frac{m}{r^2}. \quad (3)$$

This quantity may also be regarded as the *force* of attraction exerted by the mass m on a mass 1 at the distance r from m , and is therefore called briefly *the attraction at the point* where the mass 1 is situated.

262. It will be shown later (Art. 273) that the attraction of a homogeneous sphere on an external point is the same as if the mass of the sphere were concentrated at its centre. Thus, if m be the mass of the earth (here assumed as a homogeneous sphere), the attraction it exerts on a mass 1 situated at a point P above its surface, at the distance $OP=r$ from the centre O , is $=\kappa m/r^2$; and this is also the acceleration j that it would cause in any mass m' at P .

Now for points P near the earth's surface this acceleration j is known from experiments; it is the acceleration of gravity, usually denoted by g . As the radius of the earth, $r=6.37 \times 10^8$ centimetres, and its mean density $\rho=5\frac{2}{3}$, are also known, the value of the constant κ can be found from the formula

$$g = \kappa \frac{m}{r^2}$$

or

$$\kappa = \frac{3}{4} \frac{g}{\pi \rho r}$$

With $g=980$ we find in C.G.S. units

$$\kappa = \frac{1}{1.543 \times 10^7} = 0.000\,000\,0648.$$

This, then, is the force in dynes with which two masses of 1 gramme each would attract each other if concentrated at two points 1 centimetre apart.

263. Exercises.

(1) Show that the value of κ in the F.P.S. system is $\frac{1}{9.3 \times 10^8}$.

(2) When the units are so selected as to make the constant κ equal to 1, they are called *astronomical units*. Show that the astronomical unit of mass, *i.e.* a mass which when concentrated at a point produces unit acceleration at unit distance, is $= 1/\kappa$.

264. Let a mass or a system of masses be given, and let it be required to determine the *attraction at any point P* (Art. 261) produced by it. The given masses may consist of discrete particles, or they may be continuous of one, two, or three dimensions. Continuous masses must be resolved into elements; the

attraction at P produced by each element must be determined, and then all these forces must be compounded into a single resultant. This is always possible because the forces all pass through the point P .

Let dm be the element of mass situated at the point Q , P the attracted point of mass 1, $PQ=r$ the distance between them, and α, β, γ the direction cosines of r ; then the attraction at P due to dm is dm/r^2 ; its components are $\alpha dm/r^2, \beta dm/r^2, \gamma dm/r^2$; and the components of the resultant attraction R at P are

$$X = \int \frac{\alpha dm}{r^2}, \quad Y = \int \frac{\beta dm}{r^2}, \quad Z = \int \frac{\gamma dm}{r^2}, \quad (4)$$

where the integrations must be extended over the whole given mass. The resultant R itself, and its direction cosines a, b, c , are finally found from the formulæ

$$R = \sqrt{X^2 + Y^2 + Z^2}, \quad a = \frac{X}{R}, \quad b = \frac{Y}{R}, \quad c = \frac{Z}{R}. \quad (5)$$

The following examples will illustrate the process.

265: Homogeneous Circular Arc. To determine the attraction exerted by a mass distributed uniformly along a circular arc ACB (Fig. 78) of angle 2α and radius a on a mass 1 situated at the centre P of the circle, let $QQ' = ds$ be an element of the arc, $dm = \rho ds$ its mass; then

$$\kappa \frac{\rho ds}{a^2}$$

is its attraction at P , and this force has the direction PQ .

Resolving this force parallel to the bisecting radius PC of the arc and at right angles to it, it will be seen that the latter component need not be considered, since it is balanced by an equal and opposite component arising from the element situated symmetrically to QQ' with respect to PC . If $\angle CPQ = \theta$, the component along PC is $\kappa \rho ds \cos \theta / a^2$, or since $ds = a d\theta$, $\kappa \rho \cos \theta d\theta / a$. Hence the resultant attraction at P is

$$R = \frac{\kappa \rho}{a} \int_{-\alpha}^{+\alpha} \cos \theta d\theta = 2 \kappa \rho \frac{\sin \alpha}{a}. \quad (6)$$

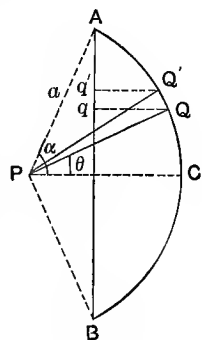


Fig. 78.

Denoting the chord AB of the given arc by c , the result can be put into the form

$$R = \kappa\rho \cdot \frac{c}{a^2}, \quad (6')$$

which might have been found directly from Fig. 78, without integration, since $ds \cos \theta = qq'$.

Formula (6') shows that, if the chord AB were covered with mass of the same density ρ as the arc, and if this mass were concentrated at the middle point C of the arc, it would produce at P the same attraction as the arc ACB .

266. Homogeneous Straight Rod. To determine the attraction at any point P produced by a mass distributed uniformly along a segment of

a straight line, AB , let $QQ' = ds$ (Fig. 79) be an element, $OQ = s$ its distance from the foot of the perpendicular $PO = p$ dropped from P on AB , and let r, θ be the polar co-ordinates of Q with respect to P as pole and PO as polar axis. Resolving the attraction $\kappa\rho ds/r^2$ of the element QQ' along PO and at right angles to it, we find the components

$$dX = \kappa\rho \frac{\cos \theta ds}{r^2}, \quad dY = \kappa\rho \frac{\sin \theta ds}{r^2}.$$

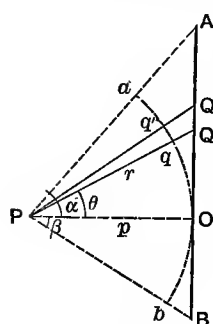


Fig. 79.

The figure gives $r = p/\cos \theta$, $s = p \tan \theta$; hence $ds = p d\theta / \cos^2 \theta$; substituting these values, we have

$$dX = \frac{\kappa p}{p} \cos \theta d\theta, \quad dY = \frac{\kappa p}{p} \sin \theta d\theta;$$

and integrating between the limits $-\beta = OPB$ and $\alpha = OPA$:

$$X = \frac{\kappa p}{p} (\sin \beta + \sin \alpha), \quad Y = \frac{\kappa p}{p} (\cos \beta - \cos \alpha). \quad (7)$$

The resultant attraction

$$R = \sqrt{X^2 + Y^2} = \frac{2\kappa p}{p} \sin \frac{\alpha + \beta}{2} \quad (8)$$

makes with PO an angle ϕ , for which we have

$$\tan \phi = \frac{Y}{X} = \frac{\cos \beta - \cos \alpha}{\sin \beta + \sin \alpha} = \tan \frac{\alpha - \beta}{2}; \quad (8)$$

hence

$$\phi = \frac{\alpha - \beta}{2}, \quad (9)$$

i.e. the attraction at P bisects the angle APB subtended at P by the rod.

267. These results might have been derived from the problem of Art. 265. For it is easy to see that the attraction exerted at P by the straight rod AB is the same as that exerted by the circular arc ab (Fig. 79) described about P with radius p and bounded by PA, PB . This will follow if it can be shown that the attraction at P of the element QQ' of the rod is equal to that of the element qq' of the arc contained between the same radii vectores. Now the attraction of QQ' is $\kappa\rho \cdot QQ'/r^2$, while that of qq' is $\kappa\rho \cdot qq'/p^2$. Projecting QQ' on the circle of radius r , we have

$$\frac{qq'}{QQ' \cos \theta} = \frac{p}{r},$$

or since the triangle POQ gives $\cos \theta = p/r$,

$$\frac{qq'}{QQ'} = \frac{p^2}{r^2};$$

which proves the proposition.

268. It has been shown in Art. 266 that the attraction at any point P exerted by a straight rod AB bisects the angle APB ; it is therefore tangent to the hyperbola passing through P and having A, B as its foci. Hence if in any plane through AB the system of confocal hyperbolas be constructed with A, B as foci, the direction of the attraction at any point P in the plane is along the tangent to the hyperbola that passes through P . These hyperbolas having everywhere the direction of the resulting attractive force, are called the **lines of force**.

An ellipse passing through P and having the same foci A, B would have the bisector of the angle APB as its normal. The confocal ellipses about A, B as foci form the so-called *orthogonal system* of the lines of force. If such an ellipse be regarded as offering a normal resistance, the point P would be kept in equilibrium under the action of the attraction of the rod and the reaction of the curve. The confocal ellipses are therefore called *equilibrium*, or *level*, *lines*, or also for a reason that will appear later **equipotential lines**.

Rotating the whole figure about AB as axis, the ellipses describe confocal ellipsoids of revolution which are *level*, or *equipotential*, surfaces.

269. Exercises.

(1) A segment AB is cut out of an infinite straight line along which mass is distributed uniformly. If the mass on the ray issuing from A be repulsive, that on the ray issuing from B (in the opposite sense), attractive, determine the resultant attraction at any point P by the method of Art. 267, and show that the lines of force are confocal ellipses, while the equipotential surfaces are confocal hyperboloids.

(2) Three rods of constant density form a triangle. Find the point at which the resultant attraction is zero.

(3) Find the attraction of a straight rod AB of constant density on a point P situated on the line AB so that $AP = a$, $BP = b$.

(4) Two rods of lengths $2a$, $2b$, and of equal constant density, are placed parallel to each other, at a distance c , so that the line joining their middle points is at right angles to them. Find their *mutual attraction*, i.e. the force required to keep them apart.

(5) Show that the attraction of a homogeneous rod of infinite length on a point at the distance p from it, is $2\kappa p/p$.

270. The formula of the last exercise (5) can be used to determine the attraction of an **infinitely long homogeneous cylinder** of finite cross-section on an external point P , by resolving the cylinder into filaments

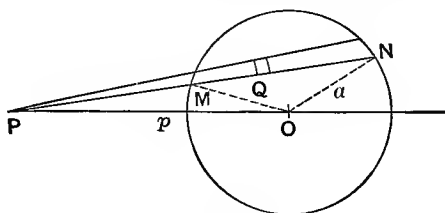


Fig. 80.

parallel to the axis. Fig. 80 represents the cross-section of the cylinder passing through P . The polar element of area at Q , $r d\theta \cdot dr$, can be regarded as the cross-section of a filament, whose attraction at P is, by Ex. (5),

$$2\kappa p \frac{r dr d\theta}{r} = 2\kappa p dr d\theta.$$

Resolving this force along and at right angles to PO , and considering that owing to symmetry the resultant R must pass through O , we have

$$R = 2 \kappa \rho \int \int \cos \theta d\theta dr.$$

If the radius vector PQ meet the surface of the cylinder at M and N , the integration with respect to r gives

$$R = 2 \kappa \rho \int MN \cos \theta d\theta.$$

With $PO = p$, we have $MN = 2\sqrt{a^2 - p^2 \sin^2 \theta}$, and the limits of the integration are $\pm \sin^{-1}(a/p)$. Hence

$$R = 2 \pi \kappa \rho \frac{a^2}{p}. \quad (10)$$

271. Homogeneous Circular Plate. To determine the attraction of a homogeneous circular area on a point P situated on the axis through the centre O at right angles to its plane at the distance $PO = p$, we may resolve the plate into ring-shaped elements of radii r and $r + dr$. The mass of such a ring is $2 \pi \rho r dr$; all points of the ring are at the same distance $\sqrt{r^2 + p^2}$ from P , and their attractions make the same angle $\phi = \tan^{-1}(r/p)$ with the axis PO . Hence the attraction of the ring is

$$2 \pi \kappa \rho \cdot \frac{r dr}{r^2 + p^2} \cos \phi = 2 \pi \kappa \rho \sin \phi d\phi,$$

since $dr = p d\phi / \cos^2 \phi = (p^2 + r^2) d\phi / p$.

Let 2α be the vertical angle of the cone that subtends the plate at P ; then

$$R = 2 \pi \kappa \rho \int_0^\alpha \sin \phi d\phi = 4 \pi \kappa \rho \sin^2 \frac{\alpha}{2}, \quad (11)$$

or, in terms of p ,

$$R = 2 \pi \kappa \rho \left(1 - \frac{p}{\sqrt{r^2 + p^2}} \right). \quad (11')$$

272. Homogeneous Spherical Shell: Geometrical Method. (a) *Attraction at an internal point P.* Let C be the centre, a the radius of the sphere (Fig. 81). A thin double cone having its vertex at P cuts the sphere in two elements, $AB = dS$, $A'B' = dS'$, which can be shown to exert equal and opposite attractions at P .

Let $PA = r$, $PA' = r'$, and let $d\omega$ denote the *solid angle of the cone* (i.e. the area it cuts out of a sphere of radius 1 described about P as centre); then $r^2 d\omega$ is the area cut out of a sphere of radius r with the same centre P . Hence the element of mass at A is $\rho \cdot r^2 d\omega / \cos PAC$, and its attraction at P is $= \kappa \rho d\omega / \cos PAC$. Similarly, the attraction of the mass element at A' is

$$= \kappa \rho r'^2 d\omega / (r'^2 \cos PA'C) = \kappa \rho d\omega / \cos PA'C.$$

These attractions are equal, since for the sphere $\angle PAC = \angle PA'C$.

The whole sphere can thus be divided up into elements exerting equal and opposite attractions at P ; the resultant attraction of the whole shell at any internal point is, therefore, zero.

273. (b) *Attraction at an external point P.* The investigation can be made similar to that for an internal point by introducing the point P' (Fig. 82), which is inverse to P with respect to the sphere, i.e. the point P' on CP for which $CP \cdot CP' = CA^2$, or putting $CP = p$, $CP' = p'$, $CA = a$, the point for which

$$pp' = a^2. \quad (12)$$

Any chord HH' through P' determines two pairs of similar triangles: CHP' and CPH , $CH'P'$ and CPH' ; for each pair has the angle at C

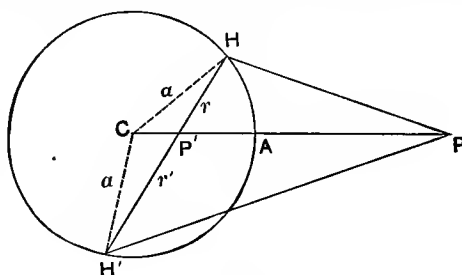


Fig. 82.

in common, and the sides including the equal angles proportional by (12), since $CH = CH' = a$. It follows that $\angle CHP' = \angle CPH$, and $\angle CH'P' = \angle CPH'$; hence, as the triangle HCH' is isosceles, the line CP bisects the angle HPH' .

With the aid of these geometrical properties it can be shown that

275. Homogeneous Spherical Shell: *Analytical method.* Let Q (Fig. 84) be any point on the sphere; $PQ = r$, $CQ = a$, $CP = p$, $\angle PCQ = \theta$. Through Q lay a plane at right angles to CP , and take as element the mass contained between this and an infinitely near parallel plane. This mass element is

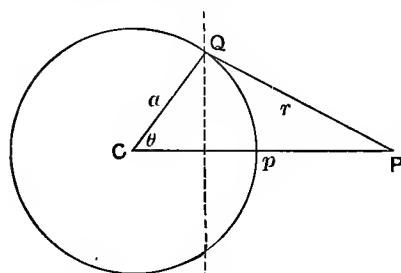


Fig. 84.

$$= \rho \cdot 2\pi a \sin \theta \cdot a d\theta,$$

and its attraction at P along CP is

$$= 2\pi\kappa\rho a^2 \sin \theta d\theta \cdot \frac{1}{r^2} \cdot \cos CPQ = 2\pi\kappa\rho a^2 \sin \theta d\theta \cdot \frac{p - a \cos \theta}{r^3}.$$

The relation between r and θ follows from the triangle CPQ , which gives

$$r^2 = a^2 + p^2 - 2ap \cos \theta,$$

hence

$$r dr = ap \sin \theta d\theta.$$

Substituting for $a \sin \theta d\theta$ and for $a \cos \theta$ their values from the last two relations, the expression for the attraction of the elementary ring becomes

$$2\pi\kappa\rho a \cdot \frac{r dr}{p} \cdot \frac{p - \frac{a^2 + p^2 - r^2}{2p}}{r^3} = \pi\kappa\rho \frac{a}{p^2} \cdot \frac{p^2 - a^2 + r^2}{r^2} \cdot dr.$$

(a) For an *internal point* P , we have $p < a$, and the limits for r are from $a - p$ to $a + p$. Hence the resultant attraction is

$$R = \pi\kappa\rho \frac{a}{p^2} \int_{a-p}^{a+p} \left(\frac{p^2 - a^2}{r^2} + 1 \right) dr = \pi\kappa\rho \frac{a}{p^2} \left[\frac{a^2 - p^2}{r} + r \right]_{a-p}^{a+p} = 0.$$

(b) For an *external point* P , we have $p > a$, and the limits are from $p - a$ to $p + a$. Hence the attraction becomes

$$R = \pi\kappa\rho \frac{a}{p^2} \left[\frac{a^2 - p^2}{r} + r \right]_{p-a}^{p+a} = 4\pi\kappa\rho \frac{a^2}{p^2} = \kappa \frac{M}{p^2}.$$

276. Exercises.

(1) Show that the attraction exerted by a right circular cone of vertical angle 2α and height h , at the vertex, is $= 2\pi\kappa\rho(1 - \cos \alpha)h$.

(2) Show that the attraction of a circular cylinder of radius a and of length l , at a point on its axis at the distance x from the nearest base, is $= 2\pi\kappa\rho[l + \sqrt{a^2 + x^2} - \sqrt{(l+x)^2 + a^2}]$.

(3) From the result of Ex. (2) show that the attraction of a cylinder extending in one sense to infinity is $= 2\pi\kappa\rho a$ at its base.

(4) Show that for a spherical shell of finite thickness, if the density be either constant or a function of the distance from the centre only, the attraction is zero at any point within the hollow of the shell, and that it is the same as if the whole mass were concentrated at the centre at any external point.

(5) Show how to find the attraction of a homogeneous spherical shell of finite thickness, at any point within the mass of the shell.

(6) Show that the attraction of a solid sphere of mass M , the density being any function of the distance from the centre, is $= \kappa M/p^2$ at any external point P , having the distance p from the centre, and that it is directly proportional to the distance of the point P from the centre when P lies within the mass.

(7) Show that the attraction of a solid homogeneous hemisphere at a point in its edge is $= \frac{2}{3}\kappa\rho a\sqrt{\pi^2 + 4}$, and that it makes with the plane of the base an angle of about $32\frac{1}{2}^\circ$.

2. THE POTENTIAL.

277. The configuration and density of any attracting masses being given, the force of attraction R exerted by these masses on a mass 1 situated at any point P can be determined both in magnitude and direction. The method illustrated on some simple examples in the preceding articles, while theoretically quite general, becomes very laborious in more complicated cases. Moreover, the required resultant R , *i.e.* the "attraction at the point P ," depends as to its magnitude and direction on the position of the point P ; and it is of interest to investigate its variation from point to point throughout space, in a similar way as was done for the example of the straight rod in Art. 268.

Investigations of this kind are greatly facilitated by the aid of a certain function called the *potential*, whose meaning and use we proceed to discuss very briefly in the following articles.

278. The attraction at the point P exerted by a single particle Q of mass m (Fig. 85) is $=m/r^2$ if the units be so selected

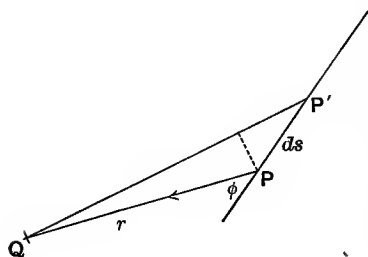


Fig. 85.

to make the constant $\kappa=1$ (Art. 262 and Art. 263, Ex. 2). (This assumption is generally made in theoretical investigations, as there is nothing to be gained by carrying the constant factor κ through all the formulæ. The factor κ can always be re-introduced when numerical results are required.)

Let the particle P be displaced through an infinitesimal distance $PP'=ds$ in any direction, and let ϕ be the angle between $QP=r$ and PP' . The element of work done by the force m/r^2 in this displacement is

$$dW = -\frac{m}{r^2} \cos \phi ds = -\frac{m}{r^2} \frac{dr}{ds} ds = \frac{d}{ds} \left(\frac{m}{r} \right) ds.$$

The quantity m/r occurring in the last expression is called the *potential* of the mass m at the point P ; it is usually denoted by V .

279. If the particle continue to move along some curve from its initial position P to some final position P_1 , the total work done by the attraction of Q is evidently

$$W = \int_P^{P_1} \frac{dV}{ds} ds = V_1 - V,$$

where $V=m/r$ is the potential at P , and $V_1=m/r_1$ is the potential at P_1 . Hence, *the difference of the potentials at any two points is equal to the work done by the attraction*, whatever may have been the path along which the displacement has taken place.

As the potential $V=m/r$ becomes zero when $r=\infty$, it appears that *the potential V_1 at any point P_1 is the work that would be*

done by the attraction on a particle of mass 1 if it were brought up to the point P_1 along any path from infinity.

The relations of Art. 278 can be written in the form

$$\frac{dV}{ds} = -\frac{m}{r^2} \cos \phi;$$

i.e. the derivative of the potential with respect to any displacement is equal to the component of the attraction in the direction of the displacement.

280. When there are given several masses $m, m', m'', \dots$, concentrated at points $Q, Q', Q'', \dots$, their potential at any point P is defined as the sum

$$V = \frac{m}{r} + \frac{m'}{r'} + \frac{m''}{r''} + \dots = \Sigma \frac{m}{r};$$

when the given masses are continuous, the sign of summation must be replaced by an integral, and we have

$$V = \int \frac{dm}{r}.$$

The fundamental properties proved in Art. 279 remain the same.

281. Let there be given a continuous mass m , referred to a rectangular system of co-ordinates. The attraction at any point $P(x, y, z)$ due to this mass has three components X, Y, Z , which can be found as follows. The attraction produced at P by an element dm at a point $Q(x', y', z')$ of the mass is dm/r^2 , where $r=PQ$, and its direction cosines are $(x'-x)/r, (y'-y)/r, (z'-z)/r$; hence its components are

$$dX = \frac{(x'-x) dm}{r^3}, \quad dY = \frac{(y'-y) dm}{r^3}, \quad dZ = \frac{(z'-z) dm}{r^3}.$$

Integrating, we find the components of the total attraction at P :

$$X = \int \frac{(x'-x) dm}{r^3}, \quad Y = \int \frac{(y'-y) dm}{r^3}, \quad Z = \int \frac{(z'-z) dm}{r^3}.$$

Differentiating the relation

$$r^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2$$

partially with respect to x , y , and z , we have

$$r \frac{\partial r}{\partial x} = -(x' - x), \quad r \frac{\partial r}{\partial y} = -(y' - y), \quad r \frac{\partial r}{\partial z} = -(z' - z).$$

Substituting these values in the above integrals, we find

$$X = - \int \frac{1}{r^2} \frac{\partial r}{\partial x} dm = \int \frac{\partial}{\partial x} \left(\frac{1}{r} \right) dm = \frac{\partial}{\partial x} \int \frac{dm}{r},$$

and similarly

$$Y = \frac{\partial}{\partial y} \int \frac{dm}{r}, \quad Z = \frac{\partial}{\partial z} \int \frac{dm}{r}.$$

As $\int \frac{dm}{r}$ is the potential V of the given mass, we have

$$X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}, \quad Z = \frac{\partial V}{\partial z};$$

i.e. *the components of the attraction at any point are the derivatives of the potential at that point in the direction of these components.* This may be regarded as a special case of the last proposition of Art. 279.

282. It is to be noticed that the proof given in the preceding article can easily be extended to the case of forces of the form (2), Art. 260. In other words, even in the case of forces not following the Newtonian law of the inverse square, but expressed by any function $f(r)$ of the distance, there exists a function corresponding to the potential of Newtonian attractions; it is called the *force function*.

We have, just as in Art. 281,

$$r^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2;$$

$$\text{hence} \quad -\frac{\partial r}{\partial x} = \frac{x' - x}{r}, \quad -\frac{\partial r}{\partial y} = \frac{y' - y}{r}, \quad -\frac{\partial r}{\partial z} = \frac{z' - z}{r}.$$

These are the direction cosines of the force $f(r)$ with which the mass m at $Q(x', y', z')$ attracts the mass 1 at $P(x, y, z)$. The components of this force are, therefore,

$$X = -f(r) \frac{\partial r}{\partial x}, \quad Y = -f(r) \frac{\partial r}{\partial y}, \quad Z = -f(r) \frac{\partial r}{\partial z}.$$

These expressions show that there exists a function

$$F(r) = - \int f(r) dr,$$

of which the components of the force at P are the partial derivatives:

$$X = \frac{\partial F(r)}{\partial x}, \quad Y = \frac{\partial F(r)}{\partial y}, \quad Z = \frac{\partial F(r)}{\partial z}.$$

283. The potential

$$V = \int \frac{dm}{r}$$

at a point P for a given system of masses is a function of the co-ordinates of the point P . If this function be known, the attraction at any point P produced by the given masses can at once be found; for the components of this attraction are

$$X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}, \quad Z = \frac{\partial V}{\partial z}.$$

Hence,

$$dV = Xdx + Ydy + Zdz.$$

If the function V be equated to any constant V_1 , the resulting equation

$$V = V_1$$

represents a surface that is the focus of all points at which the potential of the given masses has one and the same value V_1 . Such a surface is called an **equipotential surface**, or a *level*, or *equilibrium surface*.

284. Differentiating the equation of the equipotential surface, and dividing by ds and by the attraction R whose components are X, Y, Z , we find

$$\frac{\partial V}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial V}{\partial y} \cdot \frac{dy}{ds} + \frac{\partial V}{\partial z} \cdot \frac{dz}{ds} = 0.$$

The first factors in each term are the direction cosines of R , the second factors are those of a tangent to the surface; the equation expresses, therefore, the fact that *the attraction R at any point of an equipotential surface is normal to the surface.*

The attraction at any point P of an equipotential surface is, therefore, equal to dV/dn , where dn is the element of the normal at P between this and the next equipotential surface. Consequently, the attraction is inversely proportional to the distance between the successive equipotential surfaces.

Let the normal of an equipotential surface at any point P intersect the next equipotential surface at a point P' ; let the normal at P' intersect the next surface at P'' ; and so on. The elements PP' , $P'P''$, etc., will form a curve which is at every point normal to the equipotential surface passing through that point. Such a curve is called a **line of force**, since its tangent at any point indicates the direction of the resultant attraction at that point. The lines of force cut the equipotential surfaces orthogonally.

285. **Potential of a Homogeneous Spherical Shell.** (a) For an *internal point*, we may proceed similarly as in Art. 272. The element of mass cut out at A (Fig. 81) by the small cone whose solid angle is $d\omega$ is again $\rho \cdot r^2 d\omega / \cos \alpha$ if $\angle PAC = \angle PA'C = \alpha$; the corresponding potential at P is $\kappa \rho r d\omega / \cos \alpha$; similarly, the potential due to the mass at A' is $\kappa \rho r' d\omega / \cos \alpha$. Their sum is

$$\kappa \rho \frac{r + r'}{\cos \alpha} d\omega = 2 \kappa \rho a d\omega,$$

since $r + r' = 2a \cos \alpha$.

$$\text{As } \int d\omega = 2\pi, \text{ we find } V_i = 4\pi \kappa \rho a;$$

i.e. the potential has the same constant value for all points within the hollow of the shell. It follows that the attraction is zero, as found in Art. 272.

(*b*) For an *external point* P , we might also proceed geometrically, making use of the inverse point P' , as in Art. 273. But we shall use the analytical method.

Just as in Art. 275, Fig. 84, we have for the mass of the ring-shaped element $dm = \rho \cdot 2\pi a^2 \sin \theta d\theta$, or as $ap \sin \theta d\theta = r dr$, $dm = 2\pi \rho a dr/p$. Hence the element of potential is $dV = 2\pi \kappa \rho a dr/p$, which integrated between the limits from $p - a$ to $p + a$, gives

$$V_e = \frac{4\pi \kappa \rho a^2}{p} = \kappa \frac{M}{p}.$$

Hence the potential is the same as if the whole mass were concentrated at the centre.

ANSWERS.

—∞—

Pages 29-33.

(4) $\frac{1}{5}a$, where a is the side of the hexagon.

(5) At the centre of the incircle of the triangle formed by the mid-points of the sides.

(7) $\bar{x} = \bar{y} = \frac{2}{\pi} r.$

(8) Taking OA as axis of x ,

$$\bar{x} = \frac{2r}{\alpha^2} (\alpha \sin \alpha + \cos \alpha - 1), \quad \bar{y} = \frac{2r}{\alpha^2} (\sin \alpha - \alpha \cos \alpha).$$

(9) $\bar{z} = \frac{3\sqrt{2} - \log(1 + \sqrt{2})}{\sqrt{2} + \log(1 + \sqrt{2})} \cdot \frac{a}{4}, \quad \bar{y} = \frac{2\sqrt{2} - 1}{\sqrt{2} + \log(1 + \sqrt{2})} \cdot \frac{4}{3} a.$

(10) $\bar{x} = \pi a, \quad \bar{y} = \frac{4}{3} a.$

(11) $\bar{x} = \bar{y} = \frac{4}{5} a.$

(12) $\bar{x} = 0, \quad \bar{y} = \frac{cx}{s} + \frac{1}{2}y$, where $s = c(e^{\frac{x}{c}} - e^{-\frac{x}{c}})$.

(13) $\bar{x} = r \cdot \frac{\sin \theta}{\theta}, \quad \bar{y} = r \cdot \frac{1 - \cos \theta}{\theta}, \quad \bar{z} = \frac{1}{2}kr\theta.$

(14) $GC = \frac{1}{3}\sqrt{a^2 + b^2}, \quad GA = \frac{1}{3}\sqrt{a^2 + 4b^2}, \quad GB = \frac{1}{3}\sqrt{4a^2 + b^2}.$

(15) With A as origin, $\bar{x} = \frac{61}{116}a, \quad \bar{y} = \frac{63}{116}a.$

(17) $\bar{x} = \frac{1}{2} \frac{(a + a')^2 \beta + 4a'(b - \beta)\alpha}{(a + a')\beta + 2(b - \beta)\alpha}.$

(18) $\bar{x} = \frac{1}{2} \frac{a^2 + b\delta - \delta^2}{a + b - \delta} = 4.90 \text{ in.};$

first approximation $\bar{x} = \frac{1}{2} \frac{a^2 + b\delta}{a + b - \delta} = 4.62 \text{ in.};$

second approximation $\bar{x} = \frac{1}{2} \frac{a^2}{a + b} = 4.50 \text{ in.}$

$$(19) \quad \bar{x} = \frac{1}{2} \frac{a(a + 2b') - (2a - b + b')\delta}{a + b + b' - 2\delta} = 4.5 \text{ in.}$$

$$(20) \quad \bar{x} = \frac{1}{2} \frac{a_2\beta + b\alpha^2 - \alpha^2\beta}{a\beta + b\alpha - \alpha\beta}, \quad \frac{1}{2} \frac{a^2\beta + b\alpha^2}{a\beta + b\alpha - \alpha\beta}, \quad \frac{1}{2} \frac{a^2\beta}{a\beta + b\alpha}; \quad 0.32a,$$

$$0.33a, \quad 0.25a.$$

$$(25) \quad \bar{x} = -\frac{b^2c}{a^2 - b^2}.$$

(26) For a segment of a ring of angle 2α and radii r_1, r_2 , the distance of the centroid from the centre is $\bar{x} = \frac{2}{\alpha} \cdot \frac{\sin \alpha}{r_1 + r_2} \cdot \frac{r_1^2 + r_1r_2 + r_2^2}{2}$.
Hence $\bar{x} = \frac{2}{147\pi} (740 + 73\sqrt{2}) = 3.22 \text{ ft.}$; *i.e.* the centroid lies about $8\frac{1}{2}$ in. above the door.

$$(27) \quad \bar{x} = \frac{3}{5}x, \quad \bar{y} = \frac{3}{5}y.$$

$$(28) \quad \bar{x} = \frac{\pi}{2}, \quad \bar{y} = \frac{\pi}{8}.$$

$$(29) \quad \bar{x} = \frac{4}{3\pi}a, \quad \bar{y} = \frac{4}{3\pi}b.$$

$$(30) \quad \bar{x} = \frac{2}{3(\pi - 2)}a, \quad \bar{y} = \frac{2}{3(\pi - 2)}b.$$

(32) Take the vertex as origin, the axis of the cone as axis of x , and one of the bounding planes as plane of xy . Then, if a be the radius of the base, h the height, and 2α the angle at the vertex of the cone, the formulæ of Art. 40 give

$$x = f(R) = \frac{h}{a}R, \quad \sqrt{1 + [f'(R)]^2} = \sqrt{1 + \frac{h^2}{a^2}} = \csc \alpha,$$

$$S = \frac{1}{2}\phi a^2 \csc \alpha;$$

$$\text{hence} \quad \bar{x} = \frac{2}{3}h, \quad \bar{y} = \frac{2}{3}a \cdot \frac{\sin \phi}{\phi}, \quad \bar{z} = \frac{2}{3}a \cdot \frac{1 - \cos \phi}{\phi}.$$

$$(33) \text{ About } 2631 \text{ miles from the centre.}$$

(34) Regard the ice-cap as a surface mass of density δ ; let $\bar{x}_1$ be the distance of its centroid from the earth's centre, m_1 the mass of the ice-cap, m that of the earth alone, and ϕ' the polar distance of the arctic circle; then the equation of moments $(m + m_1)\bar{x} = m_1\bar{x}_1$ gives, if m_1 in the parenthesis be neglected, $\bar{x} = (m_1/m)\bar{x}_1 = \frac{1}{2}\delta \sin^2 \phi' = 0.216 \text{ mile.}$

$$(35) \text{ At the distance } \frac{1}{3}r \text{ from the centre.}$$

$$(36) \quad \bar{x} = \frac{3}{8}a.$$

$$(37) \quad \bar{x} = \frac{h}{4} \frac{r_1^3 + 2r_1r_2 + 3r_2^3}{r_1^2 + r_1r_2 + r_2^2}.$$

(38) Let V_1 be the volume of the whole pyramid, V_2 that of the top cut off, V that of the frustum; $\bar{x}_1, \bar{x}_2, \bar{x}$ the distances of their centroids from the lower base; h_1, h_2, h_3 , their heights. Then the equation of moments is $(V_1 - V_2)\bar{x} = V_1\bar{x}_1 - V_2\bar{x}_2$. By geometry, we have $V_1/V_2 = r_1^3/r_2^3$; hence, $(r_1^3 - r_2^3)\bar{x} = r_1^3\bar{x}_1 - r_2^3\bar{x}_2$; also, $\bar{x}_1 = \frac{1}{4}h_1$, $\bar{x}_2 = h + \frac{1}{4}h_2$, $h_1 - h_2 = h$, $h_1/h_2 = r_1/r_2$. By means of these relations, we find

$$(r_1^3 - r_2^3)\bar{x} = \frac{h}{4} \frac{r_1^4 - 4r_1r_2^3 + 3r_2^4}{r_1 - r_2}, \text{ or } \bar{x} = \frac{h}{4} \frac{r_1^3 + 2r_1r_2 + 3r_2^3}{r_1^2 + r_1r_2 + r_2^2}.$$

$$(39) \quad \bar{x} = \frac{\frac{3}{4}(2a - h)^2}{3a - h}.$$

(40) Let A, B, C, D be the vertices of the tetrahedron, G its centroid, G_1 that of the face ABC ; let $a, b, c, d, \bar{x}, \bar{x}_1$ be the distances of these points from the plane; and let the projections of these points on the plane be denoted by A', B', C', D', G', G'_1 . Then, since $GG_1/DG_1 = 1/4$, and $GG_1/DG_1 = (\bar{x} - \bar{x}_1)/(d - \bar{x}_1)$, we have $(\bar{x} - \bar{x}_1)/(d - \bar{x}_1) = 1/4$; hence $\bar{x} = \frac{1}{4}(3\bar{x}_1 + d)$. Let E be the middle point of AB , e its distance from the plane; then, applying a similar method to the triangle ABC , we find $\bar{x}_1 = \frac{1}{3}(2e + c) = \frac{1}{3}(a + b + c)$. Hence, finally, $\bar{x} = \frac{1}{4}(a + b + c + d)$.

$$(41) \quad \bar{x} = \frac{2}{3}h.$$

$$(43) \quad \bar{y} = \frac{5}{16}y_1.$$

$$(42) \quad \bar{y} = \frac{5}{12}y_1.$$

$$(44) \quad \bar{x} = \frac{3}{8}a, \quad \bar{y} = \frac{3}{8}b, \quad \bar{z} = \frac{3}{8}c.$$

$$(45) \quad (a) \quad \bar{x} = \bar{y} = \frac{4}{3}a.$$

$$(d) \quad \bar{x} = \frac{45\pi^2 + 128}{90\pi}a.$$

$$(b) \quad \bar{x} = \frac{9\pi^2 + 16}{18\pi}a, \quad \bar{y} = \frac{5}{6}a.$$

$$(e) \quad \bar{x} = \frac{2(15\pi - 8)}{15(3\pi - 4)}a.$$

$$(c) \quad \bar{x} = \frac{2}{15}a.$$

$$(f) \quad \bar{x} = \frac{63\pi^2 - 64}{6(9\pi^2 - 16)}a.$$

(46) Take as element a hemispherical shell of radius r and thickness dr ; $\bar{x} = \frac{n+3}{2(n+4)}a$.

$$(47) \quad \bar{x} = \frac{1}{4}(H + h).$$

(48) Compare problems (40) and (5), and apply the propositions of Pappus, Arts. 30 and 42; $V = \frac{2}{3}\pi(p + q + r)A$, where A is the area of the triangle; $S = \pi[a(q + r) + b(r + p) + c(p + q)]$.

(49) Taking the axis of the cup as axis of x , let (a, b) be the centroid of cup and handle, m their mass; $(x_1, 0)$ the centroid of the water whose mass can be expressed by $(m/c)x_1$; where c is a constant. Then the co-ordinates x, y of the centroid of cup, handle, and water together fulfil the equation $(a+c)y^2 - bxy - 2bcy + b^2c = 0$, which represents a hyperbola.

(50) Taking the axis of z parallel to the axis of the cylinder, and the origin in the line of intersection of the bases, we have $V = \iint z dx dy$, or if ϕ be the angle of inclination of the bases :

$$V = \tan \phi \iint y dx dy = \tan \phi \cdot \bar{y} \iint dx dy.$$

(51) Apply (50) twice.

Page 35.

(1) 300 000 F.P.S. units.

(3) 71 600.

(2) $34\frac{1}{11}$ miles per hour.

Page 40.

(1) 6.4×10^8 poundals $= 8.9 \times 10^9$ dynes.

(3) 0.14.

(2) 4.5 pounds.

(5) 60.

Pages 51-53.

(4) $100\sqrt{7}$; $\tan^{-1}\frac{1}{5}\sqrt{3}$.

(7) $10\sqrt{2}(\sqrt{3}-1)$; $20(\sqrt{3}-1)$.

(9) $Q = \frac{1}{2}(-P + \sqrt{4R^2 - 3P^2})$.

(10) $R = 569$; angle with horizon $= 99^\circ 27'$.

(12) Twice the focal distance.

(13) $124^\circ 12'.5$.

(14) 90° .

(15) (a) $\sqrt{2} \cdot W$; (b) $2W \cos \frac{1}{2}(\pi/2 - \theta)$.

(18) $2W \cos \frac{1}{2}(\pi/2 + \alpha)$, etc.

(19) (a) $\alpha = 30^\circ$, $\beta = 120^\circ$, $\gamma = 30^\circ$; (b) impossible.

(21) $T = \frac{9}{16}W$, nearly; $A = C = 0.86W$, $B = 0.675W$.

$$(22) \quad T = \frac{1}{8}\sqrt{3}W; \quad A = C = \frac{1}{8}\sqrt{3}W; \quad B = W.$$

$$(23) \quad \delta + 3\beta = \pi, \quad 30^\circ < \beta < 60^\circ; \quad \beta = 60^\circ.$$

$$(24) \quad \text{Resolve the components } P_1, P_2 \text{ along the bisectors of } \theta.$$

$$(25) \quad \frac{1}{2}F \tan \alpha; \quad 13.4, \quad 35.0, \quad 107.2, \quad 572, \quad \infty \text{ pounds.}$$

Pages 56-58.

$$(2) \quad R = 6, \text{ and acts along } 5.$$

$$(3) \quad T = W \cdot a/c, \quad P = W \cdot b/c.$$

$$(4) \quad AC \text{ must bisect the angle } BCW.$$

$$(5) \quad R^2 = A^2 + B^2 + C^2 + 2BC \cos \alpha + 2CA \cos \beta + 2AB \cos \gamma.$$

$$(6) \quad \text{Compare Arts. 11, 12.}$$

(7) The sum of their moments must vanish for two points in the plane not in line with their point of intersection.

$$(10) \quad P = \frac{4}{3}W; \quad T = \frac{5}{4}W.$$

$$(11) \quad T = W; \quad P = 0.89W \text{ along the bisector of } \angle BCW.$$

(12) $P = W \sin(\alpha + \beta) \sin \beta$ becomes a maximum for $\beta = (\pi - \alpha)/2$, i.e. when the sail bisects the angle between boat and wind.

$$(13) \quad W \frac{\sin \beta}{\sin(\alpha + \beta)}; \quad W \frac{\sin \alpha}{\sin(\alpha + \beta)}.$$

(14) Tension in AB and $CD = W \cdot l/d$, tension in $BC = W \cdot (c-l)/2d$, where $d = \sqrt{l^2 - \frac{1}{4}(c-l)^2}$.

(15) The resultant acts along the diameter through A , and is in magnitude equal to the perimeter.

$$(16) \quad P(1 + \sqrt{2}).$$

(17) (a) $W \cos \theta, W \sin \theta$; (b) $W \tan \theta, W/\cos \theta$; (c) $W \sin \theta/\cos \alpha, W \cos(\theta + \alpha)/\cos \alpha$.

(20) Produce BO to the intersection D with the circumscribed circle; then DA is equal and parallel to the resultant of OA, OB . $DAO'C$ is a parallelogram; hence $DA = CO'$.

Pages 67-68.

(1) Take moments about the fulcrum. The distance of this point from the end carrying the mass 12 is (a) $3\frac{6}{13}$ ft.; (b) 4 ft.

(2) (a) $A = 12\frac{2}{3}$ tons, $B = 11\frac{1}{3}$ tons; (c) $18\frac{1}{6}$, $18\frac{5}{6}$.

(3) (a) $P = W$; (b) $P = (1 + \sqrt{2})W$.

(6) (a) 19.4 tons and 21.1 tons; (b) 30.5, 9.9.

(8) Let α be the angle subtended at the centre by the side 12, and θ the angle at which the diagonal 13 is inclined to the horizon; then

$$\tan \theta = \frac{W_3 - W_1}{W_4 - W_2} \csc \alpha + \cot \alpha.$$

(9) $x = F_2 l \sin \alpha_2 / (F_1 \sin \alpha_1 + F_2 \sin \alpha_2)$.

Page 75.

(1) $C = 0.8$, $D = 1.1$, $E = 5.8$; $AB = 3.9$, $BC = 3.6$, $CD = 3.5$, $DE = 3.7$, $EF = 5.8$; reaction at $A = 3.9$, at $F = 5.8$.

(2) $H = 69.4$, $T = 73.8$.

(3) $c = \sqrt{\frac{x^3}{6(s-x)}}$; $H = wc$.

Pages 90-92.

(1) $T = 7.68$, $A = 9.76$, $E = 12.80$ pounds.

(2) $T = 2mW$, $A = \sqrt{4m^2 - 2m + 1}W$, where $m = c/l$.

(3) The three forces W , T , A must pass through a point; $\cos \phi = 2\sqrt{\frac{1}{3}(1 - m^2)}$, where $m = l/b$; $T = W \sec \phi$, $A = W \tan \phi$.

(4) $T = \frac{1}{2}W \cos \theta / \sin(\theta - \phi)$.

(5) $A_x = B = (c/l) \cot \alpha W$, $A_y = W$.

(6) $A_x = (c/2l) \sin 2\alpha W$, $A_y = [1 - (c/l) \cos^2 \alpha]W$, $B = (c/l) \cos \alpha W$; the thrust A_x in this case is to that in Ex. (5) as $\sin^2 \alpha$ is to 1.

(7) $A = W$, $C = D = (l/a) \cos \theta W$.

(8) $x = am$, $A = \sqrt{m^2 - 1}W$, $C = mW$, where $m = (l/a)^{\frac{1}{3}}$.

(9) $B = \frac{1}{2}(3w + W) \tan \alpha$, $A = (3w + W) \sqrt{\frac{1}{4} \tan^2 \alpha + 1}$.

(10) $\cos \theta = \frac{1}{3}(m + \sqrt{m^2 + 32})$, where $m = l/a$.

Page 101.

(1) The horizontal and vertical components of the reaction at C are $C_x = -\frac{h}{a+b}W$, $C_y = \frac{1}{2}\frac{b-a}{a+b}W$, where h is the perpendicular distance of C from AB , and a , b are the segments into which this perpendicular divides AB .

Pages 112-113.

(1) (a) $P = (\sin \theta + \mu \cos \theta)W = \frac{\sin(\phi + \theta)}{\cos \phi}W$; (b) $P = 0$, or $P < 2W \sin \theta$; (c) if P act up the plane, $P < \frac{\sin(\phi + \theta)}{\cos \phi}W$; if P act down the plane, $P < \frac{\sin(\phi - \theta)}{\cos \phi}W$.

(2) 2 tons.

$$(3) P = \frac{\sin \phi}{\cos(\phi - \alpha)}W.$$

$$(4) (a) P = \frac{\sin(\theta - \phi)}{\cos(\alpha + \phi)}W; (b) P = \frac{\sin(\theta + \phi)}{\cos(\alpha - \phi)}W.$$

$$(5) \theta = \frac{\pi}{2} - 2\phi.$$

$$(6) \theta = \tan^{-1} \frac{\mu W + \mu' W'}{W + W'}.$$

Pages 145-146.

(3) (a) 7500; (b) 3 160 425 000.

(5) 150 ft.-pounds.

(4) 18 000 ft.-pounds.

Pages 168-169.

(2) The centre of the inscribed circle.

$$(3) \kappa \rho \frac{a-b}{ab}.$$

$$(4) \frac{2\kappa\rho}{c} (\sqrt{c^2 + (a+b)^2} - \sqrt{c^2 + (a-b)^2}).$$

MATHEMATICAL TEXT-BOOKS

PUBLISHED BY

MACMILLAN & CO.

Mechanics.

RIGID DYNAMICS.

An Introductory Treatise. By W. STEADMAN ALDIS, M.A. \$1.00.

ELEMENTARY APPLIED MECHANICS.

Part II. Transverse Stress. By T. ALEXANDER and A. W. THOMPSON.
8vo. \$2.75.

EXPERIMENTAL MECHANICS.

A Course of Lectures delivered to the Royal College of Science for Ireland. By Sir R. S. BALL, LL.D., F.R.S. Second Edition. With Illustrations. 12mo. \$1.50.

Teachers of experimental mechanics can have no better assistant than this book, for the lectures are happy in their illustrations, clear in their demonstrations, and admirable in their arrangement. — *Manchester Guardian*.

LESSONS ON THERMODYNAMICS.

By R. E. BAYNES, M.A. 12mo. \$1.90.

WORKS BY W. H. BESANT, D.Sc.,
Lecturer of St. John's College, Cambridge.

A TREATISE ON HYDROMECHANICS.

Fifth Edition, revised. Part I. Hydrostatics. 12mo. \$1.25.

A TREATISE ON DYNAMICS. \$1.75.

ELEMENTARY HYDROSTATICS. 16mo. \$1.00.

ELEMENTS OF DYNAMIC.

An Introduction to the Study of Motion and Rest in Solid and Fluid Bodies. By W. KINGDON CLIFFORD, F.R.S.

Part I. Books I.–III. 12mo. \$1.90.

Part II. Book IV. and Appendix. 12mo. \$1.75.

APPLIED MECHANICS.

An Elementary General Introduction to the Theory of Structures and Machines. By JAMES H. COTTERILL, F.R.S. 8vo. \$5.00.

The book is a valuable one for students. . . . A very admirable feature of the book is the frequent interjecting of practical problems for solution, at the close of the different sections. — *School of Mines Quarterly*.

This work constitutes a manual of applied mechanics which, in its completeness and adaptation to the requirements of the engineer student, is without a rival. — *Academy*.

ELEMENTARY MANUAL OF APPLIED MECHANICS.

By Prof. J. H. COTTERILL, F.R.S., and J. H. SLADE. 12mo. \$1.25.

GRAPHICAL STATICS.

By LUIGI CREMONA. Two Treatises on the Graphical Calculus and Reciprocal Figures in Graphical Calculus. Authorized English Translation by T. HUDSON BEARE. 8vo. \$2.25.

A TREATISE ON ELEMENTARY DYNAMICS.

For the Use of Colleges and Schools. By WILLIAM GARNETT, M.A., D.C.L. Fifth Edition, revised. \$1.50.

ELEMENTARY STATICS.

By H. GOODWIN, D.D., Bishop of Carlisle. Second Edition. 75 cents.

WORKS BY JOHN GREAVES, M.A.

A TREATISE ON ELEMENTARY STATICS.

Second Edition, revised. 12mo. \$1.90.

It can be unreservedly recommended. — *Nature*.

STATICS FOR BEGINNERS. 16mo. 90 cents.

HYDROSTATICS.

By ALFRED GEORGE GREENHILL. *In preparation*.

THE APPLICATIONS OF PHYSICAL FORCES.

By A. GUILLEMIN. Translated and Edited by J. NORMAN LOCKYER, F.R.S. With Colored Plates and Illustrations. Royal 8vo. \$6.50.

ELEMENTARY DYNAMICS OF PARTICLES AND SOLIDS.

By W. M. HICKS. 12mo. \$1.60.

ELEMENTARY MECHANICS.

Stage I. By J. C. HOROBIN, B.A. With Numerous Illustrations. 12mo. Cloth. 50 cents. *Stages II. and III. in preparation.*

THE ELEMENTS OF GRAPHICAL STATICS.

A Text-Book for Students of Engineering. By LEANDER M. HOSKINS, C.E., M.S., Prof. of Applied Mechanics, Leland Stanford Jr. University, Palo Alto, Cal.

A TREATISE ON THE THEORY OF FRICTION.

By J. H. JELLETT, B.D., late Provost of Trinity College, Dublin. \$2.25.

THE MECHANICS OF MACHINERY.

By ALEXANDER B. W. KENNEDY, F.R.S. With Illustrations. 12mo. \$3.50.

HYDRODYNAMICS.

By H. LAMB. A Treatise on the Mathematical Theory of Fluid Motion. 8vo. \$3.00.

WORKS BY THE REV. J. B. LOCK, M.A.

Fellow of Gonville and Caius College, Cambridge.

DYNAMICS FOR BEGINNERS. 16mo. \$1.00.

Mr. Lock writes like one whose heart is in his work. He shows rare ingenuity in smoothing the difficulties in the path of the beginner, and we venture to predict that his book will meet with a wide and hearty welcome among practical teachers. — *Athenæum*.

ELEMENTARY STATICS. 16mo. \$1.10.

The whole subject is made interesting from beginning to end, and the proofs of the various propositions are very simple and clear. We have no doubt that the book will be appreciated by all who have an opportunity of judging of its merits.—*Nature*.

MECHANICS FOR BEGINNERS.

Part I. Mechanics of Solids. 90 cents.

An introductory book, which, it is hoped, may be found useful in schools and by students preparing for the elementary stage of the Science and Art and other public examinations.

ELEMENTARY HYDROSTATICS. *In preparation.*

WORKS BY S. L. LONEY, M.A.,
Fellow of Sidney Sussex College, Cambridge.

A TREATISE ON ELEMENTARY DYNAMICS.

New and Enlarged Edition. 12mo. \$1.90.

Solutions of the Examples contained in the Above. *In the Press.*

THE ELEMENTS OF STATICS AND DYNAMICS.

Part I. Elements of Statics. \$1.25. Part II. Elements of Dynamics. \$1.00. Complete in one volume. 16mo. \$1.90.

AN ELEMENTARY TREATISE ON KINEMATICS AND DYNAMICS.

By JAMES GORDON MACGREGOR, M.A., D.Sc., Munro Professor of Physics, Dalhousie College, Halifax. 12mo. \$2.60.

The problems and exercises seem to be well chosen, and the book is provided with a useful Index. Altogether it will repay a careful study.—*Educational Times*.

WORKS BY G. M. MINCHIN, M.A.,
Professor of Mathematics in the Royal Indian Engineering College.

A TREATISE ON STATICS.

Third Edition, corrected and enlarged. Vol. I. Equilibrium of Coplanar Forces. 8vo. \$2.25. Vol. II. Statics. 8vo. \$4.00.

UNIPLANAR KINEMATICS OF SOLIDS AND FLUIDS. 12mo. \$1.90.

A TREATISE ON ELEMENTARY MECHANICS.

For the use of the Junior Classes at the Universities, and the Higher Classes in Schools. With a collection of examples by S. PARKINSON, F.R.S. Sixth Edition. 12mo. \$2.25.

LESSONS ON RIGID DYNAMICS.

By the Rev. G. PIRIE, M.A. 12mo. \$1.50.

ELEMENTARY STATICS.

By G. RAWLINSON, M.A. Edited by E. STURGES. 8vo. \$1.10.

WORKS BY E. J. ROUTH, LL.D., F.R.S.,
Fellow of the Senate of the University of London.

A TREATISE ON THE DYNAMICS OF A SYSTEM OF RIGID BODIES.

With Examples. New Edition Revised and Enlarged. 8vo.

Part I. Elementary. Fifth Edition Revised and Enlarged. \$3.75.
Part II. Advanced. \$3.75.

STABILITY OF A GIVEN STATE OF MOTION,

Particularly Steady Motion. 8vo. \$2.25.

HYDROSTATICS FOR BEGINNERS.

By F. W. SANDERSON, M.A. 16mo. \$1.10. *New edition preparing.*

Mr. Sanderson's work has many merits, and is evidently a valuable text-book, being the work of a practical and successful teacher. — *Canada Educational Monthly.*

SYLLABUS OF ELEMENTARY DYNAMICS.

Part I. Linear Dynamics. With an Appendix on the Meanings of the Symbols in Physical Equations. Prepared by the Association for the Improvement of Geometrical Teaching. 4to. 30 cents.

A TREATISE ON DYNAMICS OF A PARTICLE.

By P. G. TAIT, M.A., and W. J. STEELE. Sixth Edition, Revised. \$3.00.

WORKS BY ISAAC TODHUNTER, F.R.S.

Late Fellow and Principal Mathematical Lecturer of St. John's College.

MECHANICS FOR BEGINNERS.

With Numerous Examples. New Edition. 18mo. \$1.10. KEY. \$1.75.

A TREATISE ON ANALYTICAL STATICS.

Fifth Edition. Edited by Prof. J. D. EVERETT, F.R.S. 12mo. \$2.60.

A COLLECTION OF PROBLEMS IN ELEMENTARY MECHANICS.

By W. WALTON, M.A. Second Edition. \$1.50.

PROBLEMS IN THEORETICAL MECHANICS.

Third Edition, Revised. With the addition of many fresh Problems.
By W. WALTON, M.A. 8vo. \$4.00.

Higher Pure Mathematics.

WORKS BY SIR G. B. AIRY, K.C.B., formerly Astronomer-Royal.

ELEMENTARY TREATISE ON PARTIAL DIFFERENTIAL EQUATIONS.

With Diagrams. Second Edition. 12mo. \$1.50.

ON THE ALGEBRAICAL AND NUMERICAL THEORY OF ERRORS

of Observations and the Combination of Observations. Second Edition, revised. 12mo. \$1.75.

NOTES ON ROULETTES AND GLISSETTES.

By W. H. BESANT, D.Sc., F.R.S. Second Edition, Enlarged. \$1.25.

A TREATISE ON THE CALCULUS OF FINITE DIFFERENCES.

By the late GEORGE BOOLE. Edited by J. F. MOULTON. Third Edition. 12mo. \$2.60.

ELEMENTARY TREATISE ON ELLIPTIC FUNCTIONS.

By ARTHUR CAVLEY, D.Sc., F.R.S. *New Edition preparing.*

DIFFERENTIAL CALCULUS.

With Applications and Numerous Examples. An Elementary Treatise.

By JOSEPH EDWARDS, M.A. 12mo. \$2.75.

A useful text-book both on account of the arrangement, the illustrations, and applications and the exercises, which contain more easy examples than are usually given. — *Educational Times*.

AN ELEMENTARY TREATISE ON SPHERICAL HARMONICS

And Subjects Connected with Them. By Rev. N. M. FERRERS, D.D.,

F.R.S. 12mo. \$1.90.

WORKS BY ANDREW RUSSELL FORSYTH, M.A.

A TREATISE ON DIFFERENTIAL EQUATIONS. 8vo. \$3.75.

As Boole's "Treatise on Different Equations" was to its predecessor, Wymers', this treatise is to Boole's. It is more methodical and more complete, as far as it goes, than any that have preceded it. — *Educational Times*.

THEORY OF DIFFERENTIAL EQUATIONS.

Part I. Exact Equations and Pfaff's Problems. 8vo. \$3.75.

AN ELEMENTARY TREATISE ON CURVE TRACING.

By PERCIVAL FROST, M.A. 8vo. \$3.00.

DIFFERENTIAL AND INTEGRAL CALCULUS.

With Applications. By ALFRED GEORGE GREENHILL, M.A. 12mo. \$2.00.

APPLICATION OF ELLIPTIC FUNCTIONS.

By ALFRED GEORGE GREENHILL, M.A. 12mo. \$3.50.

AN ELEMENTARY TREATISE ON THE DIFFERENTIAL AND INTEGRAL CALCULUS.

By G. W. HEMMING, M.A. 8vo. \$2.50.

THEORY OF FUNCTIONS.

By JAMES HARKNESS, M.A., Professor of Mathematics, Bryn Mawr College, and FRANK MORLEY, A.M., Professor of Mathematics, Haverford College, Pa. *In preparation*.

INTRODUCTION TO QUATERNIONS.

With Examples. By P. KELLAND, M.A., and P. G. TAIT, M.A. Second Edition. 12mo. \$2.00.

HOW TO DRAW A STRAIGHT LINE.

A Lecture on Linkages. By A. B. KEMPE, B.A. With Illustrations. *Nature Series*. 12mo. 50 cents.

DIFFERENTIAL CALCULUS FOR BEGINNERS.

With Examples. By ALEXANDER KNOX, B.A. 16mo. 90 cents.

MESSENGER OF MATHEMATICS.

Edited by J. W. L. GLAISHER. Published Monthly. 35 cents each number.

WORKS BY THOMAS MUIR,

Mathematical Master in the High School, Glasgow.

A TREATISE ON THE THEORY OF DETERMINANTS.

With Examples. 12mo. *New Edition in Preparation*.

THE THEORY OF DETERMINANTS

In the Historical Order of its Development. Part I. Determinants in General. Leibnitz (1693) to Cayley (1841). 8vo. \$2.50.

TREATISE ON INFINITESIMAL CALCULUS.

By BARTHOLOMEW PRICE, M.A., F.R.S., Professor of Natural Philosophy, Oxford.

Vol. I. Differential Calculus. Second Edition. 8vo. \$3.75.

Vol. II. Integral Calculus, Calculus of Variations, and Differential Equations. 8vo. *Reprinting*.

Vol. III. Statics, including Attractions; Dynamics of a Material Particle. 8vo. \$4.00.

Vol. IV. Dynamics of Material Systems. Together with a Chapter on Theoretical Dynamics, by W. F. DONKIN, M.A. 8vo. \$4.50.

A TREATISE ON THE THEORY OF DETERMINANTS,

And their Applications in Analysis and Geometry. By ROBERT SCOTT, M.A. 8vo. \$3.50.

FACTS AND FORMULÆ IN PURE MATHEMATICS

And Natural Philosophy. Containing Facts, Formulæ, Symbols, and Definitions. By the late G. R. SMALLEY, B.A., F.R.A.S. New Edition by J. M'DOWELL, M.A., F.R.A.S. 16mo. 70 cents.

MATHEMATICAL PAPERS OF THE LATE REV. J. S. SMITH.

Savilian Professor of Geometry in the University of Oxford. With Portrait and Memoir. 2 vols. 4to. *In preparation*.

AN ELEMENTARY TREATISE ON QUATERNIONS.

By P. G. TAIT, M.A., Professor of Natural Philosophy in the University of Edinburgh. Third Edition, Much Enlarged. 8vo. \$5.50.

WORKS BY ISAAC TODHUNTER, F.R.S.,

Late Principal Lecturer on Mathematics in St. John's College, Cambridge.

AN ELEMENTARY TREATISE ON THE THEORY OF EQUATIONS.

12mo. \$1.80.

A TREATISE ON THE DIFFERENTIAL CALCULUS.

12mo. \$2.60. KEY. \$2.60.

A TREATISE ON THE INTEGRAL CALCULUS

And its Applications. 12mo. \$2.60. KEY. \$2.60.

AN ELEMENTARY TREATISE ON LAPLACE'S,

Lamé's and Bessel's Functions. 12mo. \$2.60.

TRILINEAR CO-ORDINATES,

And Other Methods of Modern Analytical Geometry of Two Dimensions. An Elementary Treatise. By W. ALLEN WHITWORTH, M.A. 8vo. \$4.00.

Analytical Geometry. (PLANE AND SOLID**SOLID GEOMETRY.**

By W. STEADMAN ALDIS. 12mo. \$1.50.

ELEMENTS OF PROJECTIVE GEOMETRY.

By LUIGI CREMONA, LL.D. Translated by CHARLES LEUDESORF, M.A.
8vo. \$3.25.

EXERCISES IN ANALYTICAL GEOMETRY.

By J. M. DYER, M.A. With Illustrations. 12mo. \$1.25.

A TREATISE ON TRILINEAR CO-ORDINATES,

The Method of Reciprocal Polars, and the Theory of Projections. By
the Rev. N. M. FERRERS. Fourth Edition. 12mo. \$1.75.

WORKS BY PERCIVAL FROST, D.Sc., F.R.S.

Lecturer in Mathematics, King's College, Cambridge.

AN ELEMENTARY TREATISE ON CURVE TRACING. 8vo. \$3.00.

SOLID GEOMETRY.

A New Edition, revised and enlarged, of the Treatise, by FROST and
WOLSTENHOLME. Third Edition. 8vo. \$6.00.

Hints for the Solution of Problems in the above. 8vo. \$3.00.

THE ELEMENTS OF SOLID GEOMETRY.

By R. BALDWIN HAYWARD, M.A., F.R.S. 12mo. 75 cents.

THE GEOMETRY OF THE CIRCLE.

By W. J. MCCLELLAND, M.A., Trinity College, Dublin; Head Master of
Santry School. Crown 8vo. Illustrated. \$1.60.

AN ELEMENTARY TREATISE ON CONIC SECTIONS

And Algebraic Geometry. With Examples and Hints for their Solution.
By G. HALE PUCKLE, M.A. Fifth Edition, Revised and Enlarged.
12mo. \$1.90.

WORKS BY CHARLES SMITH, M.A.

Fellow and Tutor of Sidney Sussex College, Cambridge.

AN ELEMENTARY TREATISE ON CONIC SECTIONS.

Seventh Edition. 12mo. \$1.60. KEY. \$2.60.

The best elementary work on these curves that has come under our notice. — *Academy.*

AN ELEMENTARY TREATISE ON SOLID GEOMETRY. 12mo. \$2.60.

WORKS BY ISAAC TODHUNTER, F.R.S.

Late Principal Lecturer on Mathematics in St. John's College.

PLANE CO-ORDINATE GEOMETRY,

As Applied to the Straight Line and the Conic Sections. 12mo. \$1.80.
KEY. By C. W. BOURNE, M.A. 12mo. \$2.60.

EXAMPLES IN ANALYTICAL GEOMETRY

Of Three Dimensions. New Edition, Revised. 12mo. \$1.00.

ANALYTICAL GEOMETRY FOR SCHOOLS.

By Rev. T. G. VVYAN. Fifth Edition, Revised. \$1.10.

NEW AND FORTHCOMING BOOKS

ON

PURE AND APPLIED MATHEMATICS

PUBLISHED BY

MACMILLAN & CO.

A SHORT COURSE IN THE THEORY OF DETERMINANTS.

By LAENAS GIFFORD WELD, M.A., Professor of Mathematics, State University of Iowa. \$1.90, *net*.

THE MECHANICS OF HOISTING MACHINERY:

Including Accumulators, Excavators, and Pile Drivers.

A Text-book for Technical Schools and a Guide for Practical Engineers. By Dr. JULIUS WEISBACH and Prof. GUSTAV HERMANN. With 177 illustrations. Authorized translation from the second German edition by KARL P. DAHLSTROM, M.E., Instructor of Mechanical Engineering at the Lehigh University, Pa.

AN ELEMENTARY TREATISE ON THEORETICAL MECHANICS.

By ALEXANDER ZIWET, C.E., Assistant Professor of Mathematics in the University of Michigan. Part I. Kinematics. Part II. Statics.

THE APPLICATIONS OF ELLIPTIC FUNCTIONS.

By ALFRED GEORGE GREENHILL, M.A., F.R.S., Author of "Differential and Integral Equations," etc. 8vo, \$3.00, *net*.

CO-PLANAR VECTORS AND TRIGONOMETRY.

By R. BALDWIN HAYWARD, M.A., F.R.S., Author of "The Elements of Solid Geometry," etc. \$2.00, *net*.

AN ELEMENTARY TREATISE ON PURE GEOMETRY.

With numerous Examples. By JOHN WELLESLEY RUSSELL, M.A. 12mo, \$2.60, *net*.

AN ELEMENTARY TREATISE ON MODERN PURE GEOMETRY.

By R. LACHLAN, M.A. 12mo, \$2.25, *net*.

DIFFERENTIAL CALCULUS FOR BEGINNERS.

By JOSEPH EDWARDS, M.A., formerly Fellow of Sidney Sussex College, Cambridge. Author of "A Treatise on the Differential Calculus." 16mo, \$1.10, *net*.

A TREATISE ON THE FUNCTIONS OF A COMPLEX VARIABLE.

By A. R. FORSYTH, Sc.D., F.R.S., Fellow of Trinity College, Cambridge, Author of "A Treatise on Differential Equations," etc. Royal 8vo, \$8.50, *net*.

MACMILLAN & CO., PUBLISHERS, NEW YORK.

CORNELL UNIVERSITY LIBRARY



3 1924 091 870 174

